

CALCULUS - II } Lecture Notes 3

PARTIAL DERIVATIVES - 2

* GRADIENTS & DIRECTIONAL DERIVATIVES

It is useful to combine the first partial derivatives of a function into a single vector function, which is called a gradient. At any point (x,y) , we define the gradient vector $\nabla f(x,y) = \text{grad } f(x,y)$ by:

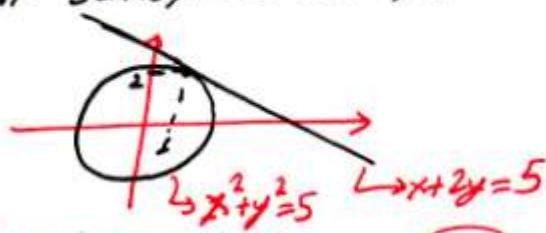
$$\nabla = i \cdot \frac{\partial}{\partial x} + j \cdot \frac{\partial}{\partial y} \text{ so that}$$

$$\nabla f(x,y) = \left(i \cdot \frac{\partial}{\partial x} + j \cdot \frac{\partial}{\partial y} \right) f(x,y) = f_1(x,y)i + f_2(x,y)j$$

*Theorem: If $f(x,y)$ is differentiable at the point (a,b) and $\nabla f(a,b) \neq 0$ then $\nabla f(a,b)$ is a normal vector to the level curve of f that passes through (a,b)

Example: If $f(x,y) = x^2 + y^2$ then $\nabla f(x,y) = 2xi + 2yj$. At point $(1,2)$, level curve is $x^2 + y^2 = 5$ and gradient vector is $\nabla f(1,2) = 2i + 4j$. This vector is the normal vector to the level curve $x^2 + y^2 = 5$. Therefore, the tangent line to the level curve is $2x + 4y = k$ that satisfies $(1,2)$:

$$2 \cdot 1 + 4 \cdot 2 = k \quad \left. \begin{array}{l} \\ k=10 \end{array} \right\} \Rightarrow \begin{aligned} 2x + 4y &= 10 \\ x + 2y &= 5 \end{aligned}$$



Example 4 If $f(x,y) = \frac{x}{x^2+y^2}$, find the gradient vector at $(1,2)$, the equation of the tangent plane at this point and the equation of the straight line tangent to the level curve at this point.

Ans $f(1,2) = \frac{1}{5}$ and level curve is $\frac{x}{x^2+y^2} = \frac{1}{5}$

$$f_1(x,y) = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad f_2(x,y) = \frac{-2xy}{(x^2+y^2)^2}$$

$$\nabla f(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2} \mathbf{i} - \frac{2xy}{(x^2+y^2)^2} \mathbf{j}$$

$$\nabla f(1,2) = \frac{3}{25} \mathbf{i} - \frac{4}{25} \mathbf{j}$$

Tangent plane: $z = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)$

$$z = \frac{1}{5} + \frac{3}{25}(x-1) - \frac{4}{25}(y-2)$$

$$25z = 5 + 3x - 3 - 4y + 8$$

$$\underline{3x - 4y - 25z + 10 = 0}$$

Line at level curve: $\frac{3}{25}x - \frac{4}{25}y + n = 0$ at $(1,2) \Rightarrow \frac{3}{25} - \frac{8}{25} + n = 0$

$$\frac{3}{25}x - \frac{4}{25}y + \frac{1}{5} = 0$$

$$n = \frac{8-3}{25} = \frac{1}{5}$$

$$\underline{3x - 4y + 5 = 0}$$

Exercise 4 Do the same for the following functions at given points

1. $f(x,y) = x^2-y^2$ at $(2,-1)$
2. $f(x,y) = \cos\left(\frac{x}{y}\right)$ at $(\pi, 4)$
3. $f(x,y) = \ln(x^2+y^2)$ at $(0,2)$

*Directional Derivatives

If f is differentiable at (a,b) and $\vec{v} = si + tj$ is a unit vector, then the directional derivative of f at (a,b) in the direction of \vec{v} is given by:

$$D_{\vec{v}} f(a,b) = \vec{v} \cdot \nabla f(a,b)$$

(if given vector \vec{v} is NOT a unit vector, take $\vec{v} = \frac{\vec{v}}{|\vec{v}|}$)

Directional derivative gives the rate of change of ~~$f(x,y)$~~ $f(x,y)$ at (a,b) in the direction of \vec{v} .

Example: Find the rate of change of

$$f(x,y) = y^4 + 2xy^3 + x^2y^2 \text{ at } (0,1)$$

measured in each of the following directions:

- a) $i+2j$
- b) $j-2i$
- c) $3i$
- d) $i+j$

Ans: $\nabla f(x,y) = (12y^3 + 2xy^2)i + (4y^3 + 6xy^2 + 2x^2y)j$

$$\nabla f(0,1) = 2i + 4j$$

a) $D_{\vec{v}} f(0,1) = \frac{i+2j}{|i+2j|} \cdot (2i+4j) = \frac{1}{\sqrt{5}} \langle 1,2 \rangle \cdot \langle 2,4 \rangle = \frac{1}{\sqrt{5}} (2+8) = \underline{\underline{2\sqrt{5}}}$

b) $D_{\vec{v}} f(0,1) = \frac{-2i+j}{|-2i+j|} \cdot (2i+4j) = \frac{1}{\sqrt{5}} \langle -2,1 \rangle \cdot \langle 2,4 \rangle = \frac{1}{\sqrt{5}} (-4+4) = \underline{\underline{0}}$

\vec{v} is perpendicular to ∇f , and so, tangent to f at its level curve.

c) $D_{\vec{v}} f(0,1) = i \cdot (2i+4j) = \langle 1,0 \rangle \cdot \langle 2,4 \rangle = 2$

rate of change is in the direction of positive x -axis.

d) $\frac{i+j}{|i+j|} (2i+4j) = \frac{1}{\sqrt{2}} \langle 1,1 \rangle \cdot \langle 2,4 \rangle = \frac{2+4}{\sqrt{2}} = 3\sqrt{2}$

rate of change is in a direction making 45° horizontal angle

Exercises Find the rate of change of the given function at the point in the specified direction.

1. $f(x,y) = 3x - 4y$ at $(0,2)$ in the direction $-2i$
2. $f(x,y) = x^2 + y^2$ at $(1,-2)$ in the direction making a (positive) angle of 60° with positive x -axis.
3. $f(x,y,z) = (y^2 + \sin z) \cdot e^{-x}$ at $(0,2,\pi)$ in the direction towards the point $(1,0,1)$

* Geometric Properties of the Gradient Vector

- (i) At (a,b) , $f(x,y)$ increases most rapidly in the direction of gradient vector $\nabla f(a,b)$. The max. rate of increase is $|\nabla f(a,b)|$.
- (ii) At (a,b) , $f(x,y)$ decreases most rapidly in the direction of gradient vector $-\nabla f(a,b)$. The max. rate of decrease is $|\nabla f(a,b)|$.
- (iii) The rate of change of $f(x,y)$ at (a,b) is zero in the directions tangent to the level curve of f that passes through (a,b) .

Examples

1. $T(x,y) = x^2 \cdot e^{-y}$ is temperature at position (x,y) . In what direction at the point $(2,1)$ does the temperature increase most rapidly? What is the rate of increase of f in that direction?

Ans $\nabla T(x,y) = 2x e^{-y} i - x^2 e^{-y} j$

$$\nabla T(2,1) = \frac{4}{e} i - \frac{4}{e} j = \frac{4}{e} (i-j)$$

At $(2,1)$, $T(x,y)$ increases most rapidly in the direction of vector $i-j$. The rate of increase in this direction is:

$$|\nabla T(2,1)| = \frac{4}{e} \sqrt{1^2 + (-1)^2} = 4\sqrt{2} e^{-1} \text{ } ^\circ\text{C/unit distance.}$$

→ A direction in the plane can be specified by a polar angle. The direction making angle ϕ with positive direction of the x -axis corresponds to the unit vector:

$$\vec{v}_\phi = \cos \phi i + \sin \phi j$$

So, the directional derivative of f at (x,y) in this direction is:

$$\begin{aligned} D_\phi f(x,y) &= D_{\vec{v}_\phi} f(x,y) = \vec{v}_\phi \cdot \nabla f(x,y) \\ &= f_1(x,y) \cos \phi + f_2(x,y) \sin \phi \end{aligned}$$

2. Find the second directional derivative of $f(x,y)$ in the direction making angle ϕ with positive x -axis.

Ans $D_\phi f(x,y) = (\cos \phi i + \sin \phi j) \nabla f(x,y) = f_1(x,y) \cos \phi + f_2(x,y) \sin \phi$

$$\begin{aligned} D_\phi^2 f(x,y) &= D_\phi (D_\phi f(x,y)) = (\cos \phi i + \sin \phi j) \cdot \nabla (f_1(x,y) \cos \phi + f_2(x,y) \sin \phi) \\ &= (f_{11}(x,y) \cos \phi + f_{12}(x,y) \sin \phi) \cos \phi \\ &\quad + (f_{21}(x,y) \cos \phi + f_{22}(x,y) \sin \phi) \sin \phi \\ &= f_{11}(x,y) \cos^2 \phi + 2 f_{12}(x,y) \cos \phi \sin \phi + f_{22}(x,y) \sin^2 \phi \end{aligned}$$

3. The height of land (in km.) at any point (x,y) is given by

$$h(x,y) = \frac{20}{3+x^2+2y^2}, \text{ where } (x,y) \text{ is coordinates. Hiker is at } (3,2).$$

- a) What is the direction of flow of the stream at $(3,2)$ on the hiker's map? How fast is the stream descending at her location?
- b) Find the equation of the path of the stream on the hiker's map.

Ans $\nabla h(x,y) = \frac{-20}{(3+x^2+2y^2)^2} \cdot (2x\mathbf{i} + 4y\mathbf{j})$

$$\nabla h(3,2) = -\frac{1}{20} (16\mathbf{i} + 8\mathbf{j}) = -\frac{1}{10} (13\mathbf{i} + 4\mathbf{j})$$

Stream is flowing in the direction $-\nabla h(3,2) \triangleq \underline{\underline{3\mathbf{i} + 4\mathbf{j}}}$
 with descending rate $|\nabla h(3,2)| = \frac{1}{10} \cdot \sqrt{3^2 + 4^2} = \underline{\underline{0.5}}$.

b) If the vector $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$ is tangent to the path of the stream at point (x,y) on the path then $d\mathbf{r}$ is parallel to $\nabla h(x,y)$. Then,

$$\frac{dx}{2x} = \frac{dy}{4y} \Rightarrow \int \frac{dy}{y} = \int \frac{2dx}{x} \Rightarrow 4y = 2\ln x + b_1 c$$

$y = cx^2$ passes through $(3,2)$

$$\text{and } 2 = c \cdot 3^2 \Rightarrow c = \frac{2}{9}$$

$$\underline{\underline{9y = 2x^2}}$$

Exercises

1. In what directions at the point $(2,0)$ does the function $f(x,y) = xy$ have rate of change -1 ? Are there directions in which rate of change is -3 ? How about -2 ?

2. An ant has coordinates (x,y) which has the temperature function $T(x,y) = x^2 - 2y^2$.

a) At what rate would the ant experience the decrease of temperature if she moves from $(2,-1)$ in the direction $-\mathbf{i} - 2\mathbf{j}$?

b) In what direction should the ant at position $(2,-1)$ move if she wishes to cool off as quickly as possible?

c) Along what curve through $(2,-1)$ should the ant move in order to continue the experience maximum rate of cooling?

3. Find the second directional derivatives of $f(x,y,z) = xyz$ at $(2,3,1)$ in the direction of $\mathbf{i} - \mathbf{j} - \mathbf{k}$.

*IMPLICIT FUNCTIONS

Remember, for single variable functions, implicit differential formulas were derived as follows:

$$F(x, y(x)) = 0$$

$$F_1(x, y) + F_2(x, y) \frac{dy}{dx} = 0$$

$$\left. \frac{dy}{dx} \right|_{x=a} = - \frac{F_1(a, b)}{F_2(a, b)} \quad \text{where } F(a, b) = 0 \text{ and} \\ \text{provided } F_2(a, b) \neq 0$$

A similar situation holds for equations involving several variables. We can, for example, ask whether the equation $F(x, y, z) = 0$ defines z as a function of x and y (say $z = h(x, y)$) near some point $P_0 = (x_0, y_0, z_0)$ satisfying the equation. If so, and if F has continuous first partial derivatives near P_0 , then differentiation of $F(x, y, z) = 0$ gives

$$F_1(x, y, z) + F_3(x, y, z) \frac{dz}{dx} = 0 \quad \text{and} \quad F_2(x, y, z) + F_3(x, y, z) \frac{dz}{dy} = 0$$

so that:

$$\left. \frac{\partial z}{\partial x} \right|_{(x_0, y_0)} = - \frac{F_1(x_0, y_0, z_0)}{F_3(x_0, y_0, z_0)} \quad \text{and} \quad \left. \frac{\partial z}{\partial y} \right|_{(x_0, y_0)} = - \frac{F_2(x_0, y_0, z_0)}{F_3(x_0, y_0, z_0)}$$

provided $F_3(x_0, y_0, z_0) \neq 0$

Example 4 Consider the sphere $x^2 + y^2 + z^2 = 1$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Ans $F(x, y, z) = x^2 + y^2 + z^2 - 1$.

$$\frac{\partial z}{\partial x} = - \frac{2x}{2z} = - \frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y} = - \frac{2y}{2z} = - \frac{y}{z}.$$

Exercises Find the following derivatives for the given functions:

1. $\frac{\partial x}{\partial y}$ if $xy^3 + x^4y = 2$ 2. $\frac{\partial z}{\partial y}$ if $z^2 + xy^3 = \frac{xz}{y}$

3. $\frac{\partial x}{\partial w}$ if $x^2y^2 + y^2z^2 + z^2t^2 + t^2w^2 - wx = 0$

* TAYLOR SERIES & APPROXIMATIONS

$f(a+h, b+k)$ can be expressed as a Taylor series in powers of h and k as follows:

$$f(a+h, b+k) = \sum_{m=0}^{\infty} \sum_{j=0}^m \frac{1}{j!(m-j)!} D_1^j D_2^{m-j} f(a, b) h^j k^{m-j}$$

As for functions of one variable, the Taylor polynomial of degree n is:

$$P_n(x, y) = \sum_{m=0}^n \sum_{j=0}^m \frac{1}{j!(m-j)!} D_1^j D_2^{m-j} f(a, b) (x-a)^j (y-b)^{m-j}$$

Example Find a second degree polynomial approximation to $f(x, y) = \sqrt{x^2 + y^3}$ near the point $(1, 2)$ and use it to estimate $\sqrt{1.02^2 + 1.97^3}$.

Ans $f(x, y) = \sqrt{x^2 + y^3}$; $f(1, 2) = 3$

$$f_1(x, y) = \frac{x}{\sqrt{x^2 + y^3}}, f_1(1, 2) = \frac{1}{3}$$

$$f_2(x, y) = \frac{3y^2}{2\sqrt{x^2 + y^3}}, f_2(1, 2) = 2$$

$$f_{11}(x, y) = \frac{y^3}{(x^2 + y^3)^{3/2}}, f_{11}(1, 2) = \frac{8}{27}$$

$$f_{22}(x, y) = \frac{12x^2y + 3y^4}{4(x^2 + y^3)^{3/2}}, f_{22}(1, 2) = \frac{2}{3}$$

$$f_{12}(x, y) = f_{21}(x, y) = \frac{-3xy^2}{2(x^2 + y^3)^{3/2}}, f_{12}(1, 2) = \frac{-2}{9}$$

Thus,

$$f(1+h, 2+ks) = 3 + \frac{1}{3} h + 2ks + \frac{1}{2!} \left[\frac{8}{27} h^2 + 2 \cdot \left(-\frac{2}{9}\right) hk + \frac{2}{3} ks^2 \right]$$

setting $x=1+h$ and $y=2+ks$, we have:-

$$f(x,y) = 3 + \frac{1}{3} (x-1) + 2(y-2) + \frac{4}{27} (x-1)^2 - \frac{2}{9} (x-1)(y-2) + \frac{1}{3} (y-2)^2$$

Therefore,

$$\sqrt{(1,02)^2 + (1,97)^3} = f(1+0,02, 2-0,03)$$

$$\approx 3 + \frac{1}{3} \cdot 0,02 + 2 \cdot (-0,03) + \frac{4}{27} 0,02^2 - \frac{2}{9} 0,02 \cdot (-0,03) + \frac{1}{3} (-0,03)^2 = 2,94716$$

Example 4 Find the Taylor polynomial of degree 3 in powers of x and y to the function $f(x,y) = e^{x-2y}$

Ans The required Taylor polynomial will be the Taylor polynomial of degree 3 for e^t evaluated at $t=x-2y$:

$$\begin{aligned} P_3(x,y) &= 1 + (x-2y) + \frac{1}{2!} (x-2y)^2 + \frac{1}{3!} (x-2y)^3 \\ &= 1 + x - 2y + \frac{1}{2} x^2 - 2xy + 2y^2 + \frac{1}{6} x^3 - x^2y + 2xy^2 - \frac{4}{3} y^3 \end{aligned}$$

Exercises

1. Find a second degree polynomial approximation to $f(x,y) = \frac{\sin x}{y}$ near the point $(\frac{\pi}{2}, 1)$ and use it to estimate $\frac{\sin 10,03 + \pi}{1,005}$

2. Find the Taylor polynomials of degree 4 for the following functions in powers of x and y (around point $(0,0)$)

a) $f(x,y) = e^{x^2+2y^2}$

b) $f(x,y) = \sin(2x+3y)$ (Hint: $\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot t^{2n+1}$)

APPLICATIONS of PARTIAL DERIVATIVES

* EXTREME VALUES

→ Remember, for single variable calculus, local/absolute extreme values can occur only at points of one of the following three types:

(i) critical points: where $f'(x)=0$, or

(ii) singular points: where $f'(x)$ does NOT exist, or

(iii) endpoints of the domain of f .

→ Likewise, for a function with two variables, $f(x,y)$ can have a local or absolute extreme value at a point (a,b) in its domain only if (a,b) is

(i) a critical point of f , satisfying $\nabla f(a,b) = \vec{0}$, or

(ii) a singular point of f , where $\nabla f(a,b)$ does NOT exist, or

(iii) a boundary point of the domain of f .

Example 4

$$1) f(x,y) = x^2 + y^2$$

$$\nabla f(x,y) = 2xi + 2yj = \langle 0,0 \rangle$$

$$\begin{array}{l} 2x=0 \\ \underline{x=0} \end{array} \quad \begin{array}{l} 2y=0 \\ \underline{y=0} \end{array}$$

Min. value at $(0,0)$

Absolute min. is $\underline{\underline{f(0,0)=0}}$

$$g(x,y) = 1 - x^2 - y^2$$

$$\nabla f(x,y) = -2xi - 2yj = \langle 0,0 \rangle$$

$$\begin{array}{l} -2x=0 \\ \underline{x=0} \end{array} \quad \begin{array}{l} -2y=0 \\ \underline{y=0} \end{array}$$

Absolute max. is $\underline{\underline{f(0,0)=1}}$
at the point $(0,0)$

2) $h(x,y) = y^2 - x^2$ and $\nabla h(x,y) = -2xi + 2yj$ has critical point at $(0,0)$ but has neither a local max nor a local min. there. When a critical point is NOT an extreme point, we call this point a saddle point.

3) $f(x,y) = \sqrt{x^2 + y^2}$ and $\nabla f(x,y) = \frac{x}{\sqrt{x^2 + y^2}} i + \frac{y}{\sqrt{x^2 + y^2}} j$.

$\nabla f(x,y) = \langle 0,0 \rangle$ has NO solution ($x=y=0$ makes denominator=0) and so, has NO critical point. Then, $(0,0)$ is a singular point. $f(x,y)$ has absolute minimum at $\underline{\underline{f(0,0)=0}}$ at $\underline{\underline{(0,0)}}$.

4) Consider the function $f(x,y) = 1-x$ in the domain $x^2+y^2 \leq 1$.
 $f(x,y)$ has max. at the boundary point $(-1,0)$; $f(-1,0) = 2$
 $f(x,y)$ has min. at the boundary point $(1,0)$; $f(1,0) = 0$

2 second derivative test

Suppose that (a,b) is a critical point of $f(x,y)$ interior to the domain of f with continuous second derivatives

$$A = f_{11}(a,b) \quad B = f_{12}(a,b) = f_{21}(a,b) \quad C = f_{22}(a,b)$$

- (i) If $B^2 < AC$ and $A > 0$ then f has a local min. at (a,b) .
- (ii) If $B^2 < AC$ and $A < 0$ then f has a local max. at (a,b) .
- (iii) If $B^2 > AC$ then f has a saddle point at (a,b)
- (iv) If $B^2 = AC$, then alles möglich :)

Example 4 -

1) Find and classify the critical points of

$$f(x,y) = 2x^3 - 6xy + 3y^2$$

Answer

$$0 = f_1(x,y) = 6x^2 - 6y \Rightarrow x^2 = y \quad (i)$$

$$0 = f_2(x,y) = -6x + 6y \Rightarrow x = y \quad (ii)$$

$$\begin{array}{l} x^2 = x \\ y = 0 \end{array}$$

$$\begin{array}{l} x = 1 \\ y = 1 \end{array}$$

Critical points are: $(0,0)$ and $(1,1)$

$f_1(x,y) = 6x^2 - 6y$	$f_2(x,y) = -6x + 6y$
$f_{11}(x,y) = 12x$	$f_{22}(x,y) = 6$
$f_{12}(x,y) = -6$	

- At $(0,0)$, $A = f_{11}(0,0) = 0$; $B = f_{12}(0,0) = -6$; $C = f_{22}(0,0) = 6$
 $B^2 - AC = (-6)^2 - 0 \cdot 6 = 36 > 0$ so, $(0,0)$ is a saddle point.
- At $(1,1)$, $A = f_{11}(1,1) = 12$; $B = f_{12}(1,1) = -6$; $C = f_{22}(1,1) = 6$
 $B^2 - AC = (-6)^2 - 12 \cdot 6 = -36 < 0$ and $A = 12 > 0$. Then,
 f has a local min. at $(1,1)$ and $f(1,1) = 2 - 6 + 3 = -1$

2) Find and classify the critical points of

$$f(x,y) = xy e^{-\frac{x^2+y^2}{2}}$$

Ans $f_1(x,y) = y(1-x^2)e^{-\frac{x^2+y^2}{2}}$ $f_2(x,y) = x(1-y^2)e^{-\frac{x^2+y^2}{2}}$

$$f_{11}(x,y) = xy(x^2-3)e^{-\frac{x^2+y^2}{2}}$$

$$f_{22}(x,y) = xy(y^2-3)e^{-\frac{x^2+y^2}{2}}$$

$$f_{12}(x,y) = (1-x^2)(1-y^2)e^{-\frac{x^2+y^2}{2}}$$

$$f_1(x,y) = 0 \text{ and } f_2(x,y) = 0 \text{ implies: } y(1-x^2) = 0 \quad y=0 \text{ or } x=\pm 1$$

$$x(1-y^2) = 0 \quad x=0 \text{ or } y=\pm 1$$

Critical points are: $(0,0)$; $(1,1)$; $(1,-1)$; $(-1,1)$; $(-1,-1)$

- At $(0,0)$: $A = C = 0$, $B = 1$ and $B^2 - AC = 1 > 0 \rightarrow$ saddle point
- At $(1,1)$ and $(-1,-1)$: $A = C = \frac{-2}{e} < 0$, $B = 0$ and $B^2 - AC = \frac{-4}{e^2} < 0$
 \rightarrow local maximum values at these points.

$$f(1,1) = f(-1,-1) = e^{-1} \rightarrow \text{this is also absolute max.}$$

- At $(1,-1)$ and $(-1,1)$: $A = C = \frac{2}{e} > 0$, $B = 0$ and $B^2 - AC = \frac{-4}{e^2} < 0$
 \rightarrow local minimum values at these points.

$$f(1,-1) = f(-1,1) = -e^{-1} \rightarrow \text{this is also absolute min.}$$

3) Find the dimensions of a rectangular box of given volume V having the least possible surface area.

Ans If the dimensions are x, y and z , we want to minimize

$$S(x, y, z) = 2xy + 2yz + 2xz$$

subject to the constraint $xyz = V \Rightarrow z = \frac{V}{xy}$ and

$$S = S(x, y) = 2xy + \frac{2V}{x} + \frac{2V}{y}$$

$$\frac{\partial S}{\partial x} = 2y - \frac{2V}{x^2} \Rightarrow x^2y = V \quad \left. \begin{array}{l} x^2y = xy^2 \\ xy(x-y) = 0 \end{array} \right\}$$

$$\frac{\partial S}{\partial y} = 2x - \frac{2V}{y^2} \Rightarrow xy^2 = V \quad \left. \begin{array}{l} xy^2 = x^2y \\ xy(x-y) = 0 \end{array} \right\}$$

since $x, y > 0, x=y = V^{1/3}$ and

$$S \text{ is min. when } x=y=z = V^{1/3} \quad z = \frac{V}{xy} = V^{1/3}$$

absolute min. of S is $S(V^{1/3}, V^{1/3}, V^{1/3}) = 6V^{2/3}$

Exercises

(I) Find and classify the critical points of the given functions:

$$1. f(x, y) = x^3 + y^3 - 3xy \quad 2. f(x, y) = x \sin y \quad 3. f(x, y) = x^2 y e^{-(x^2+y^2)}$$

$$4. f(x, y) = \left(1 + \frac{1}{x}\right) \left(1 + \frac{1}{y}\right) \left(\frac{1}{x} + \frac{1}{y}\right) \quad 5. f(x, y, z) = xy + x^2 z - x^2 y - z^2$$

(II) Find the maximum and minimum values of the given functions:

$$1. f(x, y) = xy e^{-x^2-y^2} \quad 2. f(x, y) = \frac{x}{1+x^2+y^2}$$

(III) 1. Find the dimensions of the rectangular box with no tops having given volume V and the least possible total surface area of its five faces.

2. Find three positive numbers a, b , and c , whose sum is 30, and for which the expression ab^2c^3 is maximum.

*EXTREME VALUES of FUNCTIONS DEFINED on RESTRICTED DOMAINS

Examples

1) Find the maximum and minimum values of $f(x,y) = 2xy$ on the closed disk $x^2+y^2 \leq 4$

Ans $f_1(x,y) = 2y = 0$ $f_2(x,y) = 2x = 0$
 $y=0$ $x=0$

- Only critical point in the domain of $f(x,y)$ is $(0,0)$ and $f(0,0)=0$. There is NO singular point.

$B=f_{12}(x,y)=2$; $A=C=0$; $B^2-AC=4>0 \Rightarrow (0,0)$ is saddle point.

- We can express f as a function of a single variable on the circle $x^2+y^2=4$ by using convenient parametrization.

$$\begin{aligned} x &= 2\cos t & -\pi \leq t \leq \pi \\ y &= 2\sin t \end{aligned} \quad \left. \begin{array}{l} g(t) = f(x,y) = f(2\cos t, 2\sin t) \\ g(t) = 8\cos t \sin t \end{array} \right\}$$

Extreme values for $g(t)$:

$$g'(t) = -8\sin^2 t + 8\cos^2 t = 0$$

$$\sin^2 t = \cos^2 t$$

$$\underline{\sin t = 1}$$

$$\underline{\sin t = -1}$$

$$t = \frac{\pi}{4}$$

$$t = -\frac{3\pi}{4}$$

$$t = \frac{3\pi}{4}$$

$$t = -\frac{\pi}{4}$$

- $t = \frac{\pi}{4} \Rightarrow x=y=\sqrt{2}$ and $t = -\frac{3\pi}{4} \Rightarrow x=y=-\sqrt{2}$

$f(\sqrt{2}, \sqrt{2}) = f(-\sqrt{2}, -\sqrt{2}) = 4$ is max. value of f

- $t = -\frac{\pi}{4} \Rightarrow x=\sqrt{2}, y=-\sqrt{2}$ and $t = \frac{3\pi}{4} \Rightarrow x=-\sqrt{2}, y=\sqrt{2}$

$f(\sqrt{2}, -\sqrt{2}) = f(-\sqrt{2}, \sqrt{2}) = -4$ is min. value of f .

2) Find the extreme values of the function

$f(x,y) = xy \cdot e^{-(x+y)}$ on the triangular region

$T: x \geq 0, y \geq 0$ and $x+y \leq 4$.

~~Ans~~ $f_1(x,y) = xy(2-x)e^{-(x+y)} = 0 \Rightarrow x=0$ or $x=2$ or $y=0$

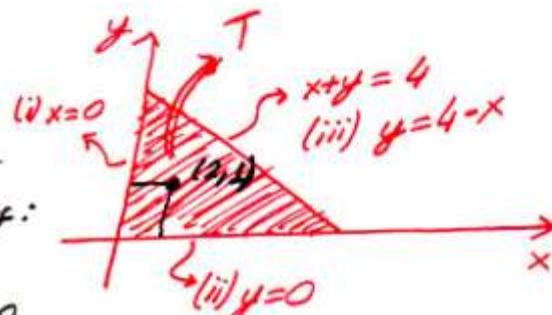
$$f_2(x,y) = x^2(1-y)e^{-(x+y)} = 0 \Rightarrow x=0 \text{ or } y=1$$

The critical points are $(0,y)$ for any y and $(2,1)$.

Only $(2,1)$ is an interior point of T

$$f(2,1) = \frac{4}{e^3} \approx 0,199$$

It is useful to graph the boundary to analyze boundary points:



On (i) and (ii), f is identically 0.

$$\text{On (iii), } g(x) = f(x, 4-x) = x^2(4-x) \cdot e^{-4} \quad 0 \leq x \leq 4$$

$$g'(x) = (8x - 3x^2) \cdot e^{-4} = 0 \Rightarrow x=0 \text{ and } x=\frac{8}{3}$$

$$g(0)=0 \quad \text{and} \quad g\left(\frac{8}{3}\right) = f\left(\frac{8}{3}, \frac{4}{3}\right) = \frac{256}{27} \cdot e^{-4} \approx 0,174 < f(2,1)$$

Therefore,

Max. of f over T is $\frac{4}{e^3}$ at $(2,1)$

Min. of f over T is 0 on line segments (i) and (ii)

Exercises Find the extreme values of the following functions in the defined domains:

1. $f(x,y) = x - x^2 + y^2$ on the rectangle $0 \leq x \leq 2, 0 \leq y \leq 1$

2. $f(x,y) = (x+y) \cdot e^{-x^2-y^2}$ on the disk $x^2 + y^2 \leq 1$

3. $f(x,y) = \sin x \cos y$ on the triangle $x \geq 0, y \geq 0$ and $x+y \leq 2\pi$

4. $f(x,y,z) = xy^2 + yz^2$ over the ball $x^2 + y^2 + z^2 \leq 1$

*LAGRANGE MULTIPLIERS

(I) maximize (or minimize) $f(x,y)$ subject to $g(x,y)=0$

→ To find candidates for points on the curve $g(x,y)=0$ at which $f(x,y)$ is extremum, we should look for critical points of the Lagrangian Function.

$$L(x,y,\lambda) = f(x,y) + \lambda \cdot g(x,y)$$

At any critical point of L , we must have:

$$0 = \frac{\partial L}{\partial x} = f_1(x,y) + \lambda \cdot g_1(x,y) \quad \left. \right\} \text{i.e. } \nabla f \parallel \nabla g$$

$$0 = \frac{\partial L}{\partial y} = f_2(x,y) + \lambda \cdot g_2(x,y)$$

$$0 = \frac{\partial L}{\partial \lambda} = g(x,y) \quad \left. \right\} \text{the constraint equation}$$

Example 11

1) Find the shortest distance from the origin to the curve $x^2y=16$

Ans: Minimize $f(x,y) = x^2 + y^2$ subject to $g(x,y) = x^2y - 16 = 0$

$$L(x,y,\lambda) = x^2 + y^2 + \lambda \cdot (x^2y - 16)$$

$$(i) 0 = \frac{\partial L}{\partial x} = 2x + 2\lambda xy = 2x(1 + \lambda y) \Rightarrow x=0 \text{ or } \lambda y = -1$$

λ inconsistent with (ii))

$$(ii) 0 = \frac{\partial L}{\partial y} = 2y + \lambda x^2 \Rightarrow 2y^2 + \lambda x^2 = 0$$

$$(iii) 0 = \frac{\partial L}{\partial \lambda} = x^2y - 16 \quad \left. \right\} \begin{aligned} & 2y^2 - x^2 = 0 \\ & x = \pm \sqrt{2}y \end{aligned} \quad \left. \right\} \begin{aligned} & x^2y = 16 \\ & 2y^3 = 16 \\ & y = 2 \end{aligned} \quad \left. \right\} x = \pm 2\sqrt{2}$$

Candidate points: $(2\sqrt{2}, 2)$; $(-2\sqrt{2}, 2)$

$$f(2\sqrt{2}, 2) = f(-2\sqrt{2}, 2) = 12$$

Min. distance of $x^2y=16$ is $\sqrt{12} = 2\sqrt{3}$ at points $(\pm 2\sqrt{2}, 2)$.

2) Find the points on the curve $17x^2 + 12xy + 8y^2 = 100$ that are closest to and farthest away the origin.

Ans Again, we want to **extremize** $x^2 + y^2$ subject to our equality constraint. The Lagrangian function is:

$$L(x, y, \lambda) = x^2 + y^2 + \lambda \cdot (17x^2 + 12xy + 8y^2 - 100)$$

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x} = 2x + \lambda \cdot (34x + 12y) \\ 0 &= \frac{\partial L}{\partial y} = 2y + \lambda \cdot (12x + 16y) \\ 0 &= \frac{\partial L}{\partial \lambda} = 17x^2 + 12xy + 8y^2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \lambda &= \frac{-2x}{34x + 12y} = \frac{-2x}{12x + 16y} \\ 12x^2 + 16xy &= 34xy + 12x^2 \\ 4/2x^2 - 3xy - 2y^2 &= 0 \text{ (i)} \\ 17x^2 + 12xy + 8y^2 &= 100 \text{ (ii)} \end{aligned}$$

$$\begin{aligned} \text{for } x=2: & y^2 + 3y - 4 = 0 \\ & (y-1)(y+4) = 0 \\ & y=1, y=-4 \\ \text{for } x=-2: & y^2 - 3y - 4 = 0 \\ & (y+1)(y-4) = 0 \\ & y=-1, y=4 \end{aligned}$$

$$25x^2 = 100$$

$$x = \pm 2$$

Candidates: $(2, 1); (-2, -1); (2, -4); (-2, 4)$

$$f(2, 1) = f(-2, -1) = 5 \rightarrow \text{min.}$$

$$f(2, -4) = f(-2, 4) = 20 \rightarrow \text{max.}$$

* **Remark:** When applying the method of Lagrange multipliers, be aware that an extreme value may occur at

- (i) a critical point of the Lagrangian
- (ii) a point where $\nabla g = 0$
- (iii) a point where ∇f or ∇g does NOT exist
- (iv) an "endpoint" of the constraint set.

3) Find min. value of $f(x, y) = y$ subject to $g(x, y) = y^3 - x^2 = 0$

Ans $L(x, y, \lambda) = y + \lambda(y^3 - x^2)$

- (i) $-2\lambda x = 0$
 - (ii) $1 + 3\lambda y^2 = 0$
 - (iii) $y^3 - x^2 = 0$
- (iii) requires $\lambda \neq 0$ and $y \neq 0$
 Then, (i) requires $x = 0$.
 But if $x = 0$, (iii) requires $y = 0$, which is a contradiction.
 NO solution (x, y, λ) .

$$\begin{aligned} \nabla g &= -2x\mathbf{i} + 3y^2\mathbf{j} = \langle 0, 0 \rangle \\ -2x &= 0 \\ x &= 0 \\ 3y^2 &= 0 \\ y &= 0 \end{aligned}$$

$$f(0, 0) = 0 \text{ is min. of } f.$$

(II) extremize $f(x, y, z)$ subject to $g(x, y, z) = 0$ and $h(x, y, z) = 0$

$\hookrightarrow (x_0, y_0, z_0, \lambda_0, \mu_0)$ is a critical point of the Lagrangian Function.

$$L(x, y, z, \lambda, \mu) = f(x, y, z) + \lambda \cdot g(x, y, z) + \mu \cdot h(x, y, z)$$

so that,

$$f_1(x, y, z) + \lambda \cdot g_1(x, y, z) + \mu \cdot h_1(x, y, z) = 0$$

$$f_2(x, y, z) + \lambda \cdot g_2(x, y, z) + \mu \cdot h_2(x, y, z) = 0$$

$$f_3(x, y, z) + \lambda \cdot g_3(x, y, z) + \mu \cdot h_3(x, y, z) = 0$$

$$g(x, y, z) = 0$$

$$h(x, y, z) = 0$$

Example Given $f(x, y, z) = xy + 2z$, find the max. and min. values of f on the circle that is the intersection of the plane: $x+y+z=0$ and sphere: $x^2+y^2+z^2=24$

$$\text{Ans} \ L(x, y, z, \lambda, \mu) = xy + 2z + \lambda(x+y+z) + \mu(x^2+y^2+z^2-24)$$

$$\begin{aligned} (i) \quad & y + \lambda + 2\mu x = 0 \\ (ii) \quad & x + \lambda + 2\mu y = 0 \\ (iii) \quad & 2 + \lambda + 2\mu z = 0 \\ (iv) \quad & x + y + z = 0 \\ (v) \quad & x^2 + y^2 + z^2 = 24 \end{aligned}$$

$$(i) - (ii):$$

$$y + 2\mu x - x - 2\mu y = 0$$

$$y - x - 2\mu(y - x) = 0$$

$$(y-x)(1-2\mu) = 0$$

$$\begin{aligned} & x=y \quad \text{OR} \quad \mu = \frac{1}{2} \\ & \text{numur} \end{aligned}$$

$$\begin{aligned} & \mu = \frac{1}{2} \Rightarrow (ii) \ x + \lambda + y = 0 \\ & (iii) \ 2 + \lambda + z = 0 \\ & \underline{\underline{x+y=2+z}} \\ & (iv) \ x + y - z = 2 \\ & \underline{\underline{-x-y-z=0}} \\ & \underline{\underline{z=-1}} \Rightarrow (iv) \ x+y=1 \end{aligned}$$

$$\hookrightarrow (v) \ x^2 + y^2 = 24 - z^2$$

$$x^2 + y^2 = 23 \quad \hookrightarrow z = -1$$

$$x^2 + y^2 + 2xy = 23 + 2xy$$

$$(x+y)^2 = 23 + 2xy$$

$$\cancel{x+y=1}$$

$$2xy = -22$$

$$xy = -11$$

$$(x-y)^2 = \cancel{x^2+y^2} - \cancel{2xy}$$

$$(x-y)^2 = 23 - 2 \cdot (-11)$$

$$(x-y)^2 = 45$$

$$\cancel{(x-y)^2 = \pm 3\sqrt{5}}$$

So, (†) and (‡) implies:

$$\begin{aligned} x+y &= 1 \\ x-y &= 3\sqrt{5} \\ \hline x &= \frac{1+3\sqrt{5}}{2} \\ y &= \frac{1-3\sqrt{5}}{2} \\ z &= -1 \end{aligned}$$

A candidate.

$$\begin{aligned} x+y &= 1 \\ x-y &= -3\sqrt{5} \\ \hline x &= \frac{1-3\sqrt{5}}{2} \\ y &= \frac{1+3\sqrt{5}}{2} \\ z &= -1 \end{aligned}$$

A candidate.

At P_1 and P_2 , we have: $f(x,y,z) = xy + 2z = -11 - 2 = -13$ min. off

$$\begin{array}{l} \cancel{x=y} \Rightarrow \text{(iv)} \ x+x+z=0 \\ \cancel{z=-2x} \end{array} \quad \text{(v)} \quad \begin{aligned} x^2+x^2+4x^2 &= 24 \\ 6x^2 &= 24 \end{aligned}$$

$x = \pm 2$ gives candidates:

$$P_3(2, 2, -4) \text{ and } P_4(-2, -2, 4)$$

At P_3 and P_4 , we have: $f(x,y,z) = f(2, 2, -4) = -4$ and

$$f(x,y,z) = f(-2, -2, 4) = 12 \rightarrow \text{max. of } f$$

Min. value -13 at P_1 and P_2 , max. value 12 at P_4 .

Exercises

- Maximize x^3y^5 subject to $x+y=8$.
- Find the minimum distance from the origin to the plane $x+2y+3z=3$.
- Find the maximum and minimum values of $f(x,y,z)=xyz$ on the sphere $x^2+y^2+z^2=12$.
- Find the maximum and minimum values of $f(x,y,z)=4-z$ on the ellipse formed by the intersection of the cylinder: $x^2+y^2=8$ and the plane $x+y+z=1$.