

CALCULUS - II

Lecture Notes 2

PARTIAL DERIVATIVES - 1

* Functions of Several Variables

→ Suppose D is a set of n -tuples of real numbers (x_1, x_2, \dots, x_n) .
 A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \dots, x_n)$$

to each element in D . The set D is the function's **domain**.
 The set of w -values taken on by f is the function's **range**.
 The symbol w is the **dependent variable** of f , and f is said to be a function of the n **independent variables** x_1 to x_n .

Ex: The value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(3, 0, 4)$ is:

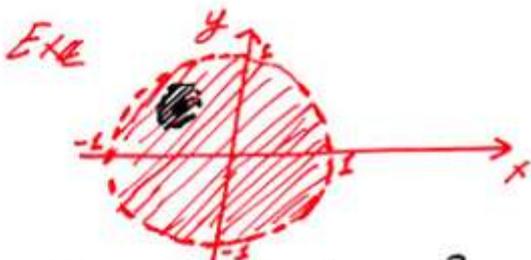
$$f(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5$$

Ex: Some functions with their domains and ranges are given:

Function	Domain	Range
$w = \sqrt{y^2 - x^2}$	$y \geq x^2$	$[0, \infty)$
$w = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$w = \sin(xy)$	\mathbb{R}^2	$[-1, 1]$
$w = \sqrt{x^2 + y^2 + z^2}$	\mathbb{R}^3	$[0, \infty)$
$w = \frac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	$z > 0$	$(-\infty, \infty)$

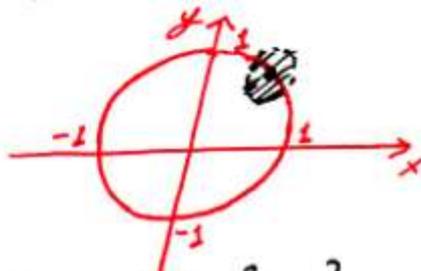
↳ A point (x_0, y_0) in a region R in the xy -plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R . A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that

The interior points of a region, as a set, lie in R . make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.



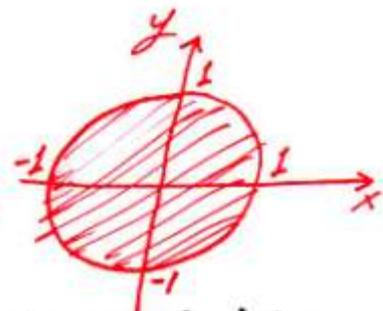
$$\{(x, y) \mid x^2 + y^2 < 1\}$$

Open unit disc



$$\{(x, y) \mid x^2 + y^2 = 1\}$$

Boundary of unit disc



$$\{(x, y) \mid x^2 + y^2 \leq 1\}$$

↳ A region in the plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is NOT bounded.

EXE $f(x, y) = \sqrt{y - x^2}$ has domain $y \geq x^2$. The domain is closed since it contains its boundary $y = x^2$. The domain is unbounded since y can take arbitrarily large values. (There is no disk of fixed radius that contains the region $y \geq x^2$)

↳ A point (x_0, y_0, z_0) in a region R in space is an **interior point** of a solid ball that lies entirely in R . We can also define boundary point, interior, boundary, open set and closed set as before by replacing disks with solid ball.

* Limits and Continuity in Higher Dimensions

↳ We say that a function $f(x,y)$ approaches the limit L as (x,y) approaches (x_0,y_0) , and write:

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$$

↳ The following rules hold if L, M and k are real numbers and $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$ and $\lim_{(x,y) \rightarrow (x_0,y_0)} g(x,y) = M$

$$1. \lim_{(x,y) \rightarrow (x_0,y_0)} (f(x,y) \pm g(x,y)) = L \pm M$$

$$2. \lim_{(x,y) \rightarrow (x_0,y_0)} (k \cdot f(x,y)) = k \cdot \lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = k \cdot L$$

$$3. \lim_{(x,y) \rightarrow (x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$$

$$4. \lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \text{ if } M \neq 0$$

$$5. \lim_{(x,y) \rightarrow (x_0,y_0)} (f(x,y))^{r/s} = L^{r/s} \text{ provided } L^{r/s} \in \mathbb{R}$$

Examples: Find the following limits if they exist.

$$1. \lim_{(x,y) \rightarrow (2,-3)} \left(\frac{1}{x} + \frac{1}{y} \right)^2$$

$$2. \lim_{(x,y) \rightarrow (0,0)} \frac{e^y \sin x}{x}$$

$$3. \lim_{(x,y) \rightarrow (1,1)} \frac{x^2 - 2xy + y^2}{x - y}$$

$$4. \lim_{(x,y) \rightarrow (1,1)} \frac{xy - y - 2x + 2}{x - 1}$$

$$5. \lim_{(x,y) \rightarrow (0,0)} \frac{x - y + 2\sqrt{x - 2\sqrt{y}}}{\sqrt{x} - \sqrt{y}}$$

$$6. \lim_{(x,y) \rightarrow (4,3)} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$$

Answers

$$1. \lim_{(x,y) \rightarrow (2,-3)} \left(\frac{1}{x} + \frac{1}{y} \right)^2 = \left(\frac{1}{2} - \frac{1}{3} \right)^2 = \left(\frac{3-2}{6} \right)^2 = \frac{1}{36}$$

$$2. \lim_{(x,y) \rightarrow (0,0)} \frac{e^y \cdot \sin x}{x} = \lim_{y \rightarrow 0} e^y \cdot \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \cdot 1 = 1$$

by L'Hospital
 $= \lim_{x \rightarrow 0} \frac{\cos x}{1} = \cos 0 = 1$

$$3. \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - 2xy + y^2}{x - y} = \lim_{(x,y) \rightarrow (1,1)} \frac{(x-y)^2}{x-y} = \lim_{(x,y) \rightarrow (1,1)} (x-y) = 1-1=0$$

$= \frac{0}{0}$, factorize the limit

$$4. \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq 1}} \frac{xy - y - 2x + 2}{x-1} = \lim_{(x,y) \rightarrow (1,1)} \frac{y(x-1) - 2(x-1)}{x-1} = \lim_{(x,y) \rightarrow (1,1)} \frac{(x-1)(y-2)}{(x-1)}$$

$= \frac{0}{0}$, factorize the limit

$$= \lim_{(x,y) \rightarrow (1,1)} (y-2) = 1-2 = -1$$

$$5. \lim_{\substack{(x,y) \rightarrow (0,0) \\ x \neq y}} \frac{x-y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y) \rightarrow (0,0)} \frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y}) + 2(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}}$$

$= \frac{0}{0}$, factorize the limit

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{(\sqrt{x}-\sqrt{y})[\sqrt{x}+\sqrt{y}+2]}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y) \rightarrow (0,0)} [\sqrt{x}+\sqrt{y}+2] = \sqrt{0}+\sqrt{0}+2=2$$

$$6. \lim_{\substack{(x,y) \rightarrow (4,3) \\ x \neq y+1}} \frac{\sqrt{x} - \sqrt{y+1}}{x-y-1} = \lim_{(x,y) \rightarrow (4,3)} \frac{(\sqrt{x}-\sqrt{y+1})(\sqrt{x}+\sqrt{y+1})}{(x-y-1)(\sqrt{x}+\sqrt{y+1})}$$

$= \frac{0}{0}$, multiply by conjugate

$$= \lim_{(x,y) \rightarrow (4,3)} \frac{x - (y+1)}{(x-y-1)(\sqrt{x}+\sqrt{y+1})} = \lim_{(x,y) \rightarrow (4,3)} \frac{1}{\sqrt{x}+\sqrt{y+1}}$$

$$= \frac{1}{\sqrt{4}+\sqrt{3+1}} = \frac{1}{2+2} = \frac{1}{4}$$

Exercises Find the following limits if they exist.

1. $\lim_{(x,y) \rightarrow (\pi/2, 0)} \frac{\cos y + 1}{y - \sin x}$

2. $\lim_{(x,y) \rightarrow (1,0)} \frac{x \cdot \sin y}{x^2 + 1}$

3. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - y^2}{x - y}$

4. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq 1}} \frac{xy - y - 2x + 2}{x - 1}$

5. $\lim_{\substack{(x,y) \rightarrow (2,-4) \\ y \neq -4, x \neq x^2}} \frac{y+4}{x^2y - xy + 4x^2 - 4x}$

6. $\lim_{\substack{(x,y) \rightarrow (2,0) \\ 2x-y \neq 4}} \frac{\sqrt{2x-y} - 2}{2x-y-4}$

→ **Two Path Test for Nonexistence of a limit**

If a function $f(x,y)$ has different limits among two different paths as (x,y) approaches (x_0, y_0) , then $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x,y)$ does NOT exist.

Examples Find the following limits if they exist.

1. $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$, $f(x,y) = \frac{2x^2y}{x^4 + y^2}$

2. $\lim_{(x,y) \rightarrow (0,0)} -\frac{x}{\sqrt{x^2 + y^2}}$

Answer

1. $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \frac{0}{0}$. The point $(0,0)$ satisfies the path $y = kx^2$.

There are infinitely many paths that pass through $(0,0)$ but this path cancels out x^4 .

$$\begin{aligned} \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx^2}} \frac{2x^2y}{x^4 + y^2} &= \lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 \cdot kx^2}{x^4 + (kx^2)^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{2kx^4}{x^4(1+k^2)} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{2k}{1+k^2} = \frac{2k}{1+k^2} \end{aligned}$$

But this is a function of k and is NOT unique.

(i.e. $k=1 \Rightarrow L=1$, $k=2 \Rightarrow L=\frac{4}{5}$ etc.). Therefore, limit does NOT exist.

2. $\lim_{(x,y) \rightarrow (0,0)} -\frac{x}{\sqrt{x^2+y^2}} = \frac{0}{0}$. Try the path $y=kx$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx}} -\frac{x}{\sqrt{x^2+y^2}} = \lim_{(x,y) \rightarrow (0,0)} -\frac{x}{\sqrt{x^2+k^2x^2}} = \lim_{(x,y) \rightarrow (0,0)} -\frac{x}{|x|\sqrt{1+k^2}}$$

$$= \frac{\pm 1}{\sqrt{1+k^2}}. \text{ Therefore, limit does NOT exist.}$$

Exercises

1. $\lim_{(x,y) \rightarrow (0,0)} \frac{x-y}{x+y}$

2. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2-y}$

↳ The Sandwich Theorem

For functions of two variables, if $g(x,y) \leq f(x,y) \leq h(x,y)$ for all $(x,y) \neq (x_0,y_0)$ in a disc centered at (x_0,y_0) and if g and h have the same limit L as $(x,y) \rightarrow (x_0,y_0)$, then

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$$

Example Find $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x}$ if it exists.

Ans We know that $|\sin(\theta)| \leq 1$. Then;

$$-1 \leq \frac{\sin(xy)}{xy} \leq 1$$

$$-y \leq \frac{\sin(xy)}{x} \leq y$$

Therefore; $\lim_{(x,y) \rightarrow (0,0)} -y = \lim_{(x,y) \rightarrow (0,0)} y = \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x} = 0$

Exercise Find $\lim_{(x,y) \rightarrow (0,0)} x \cdot \cos \frac{1}{y}$ if it exists.

* Continuity

A function $f(x,y)$ is **continuous** at the point (x_0, y_0) if

$$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x,y) = f(x_0, y_0)$$

A function is **continuous** if it is continuous at every point of its domain.

Example

$$f(x,y) = \begin{cases} \frac{2xy}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Show that $f(x,y)$ is continuous at every point except the origin.

Ans The function f is continuous at any point $(x,y) \neq (0,0)$ because its values are given by a rational function of x and y .

At $(0,0)$, we need to show that limit is NOT 0.

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx}} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} \frac{2xkx}{x^2 + (kx)^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{2k}{1+k^2} = \frac{2k}{1+k^2}$$

Therefore, limit of $f(x,y)$ does NOT exist at $(0,0)$.

Exercise Let $f(x,y) = \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$ for $(x,y) \neq (0,0)$

Can you define $f(0,0)$ so that f is continuous everywhere?

(Hint: Find $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ if it exists. If it doesn't exist,

answer is NO! If limit is L , answer is: $f(x,y) = \begin{cases} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} & (x,y) \neq (0,0) \\ L & (x,y) = (0,0) \end{cases}$

* PARTIAL DERIVATIVES

The first partial derivatives of the function $f(x,y)$ with respect to the variables x and y are the functions $f_1(x,y)$ and $f_2(x,y)$, given by:

$$f_1(x,y) = \frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_2(x,y) = \frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y} = \lim_{k \rightarrow 0} \frac{f(x,y+k) - f(x,y)}{k}$$

provided these limits exist.

↳ Values of partial derivatives:

$$\left. \frac{\partial z}{\partial x} \right|_{(a,b)} = \left(\frac{\partial f(x,y)}{\partial x} \right) \Big|_{(a,b)} = f_1(a,b)$$

$$\left. \frac{\partial z}{\partial y} \right|_{(a,b)} = \left(\frac{\partial f(x,y)}{\partial y} \right) \Big|_{(a,b)} = f_2(a,b)$$

Examples

1. $f(x,y) = x^2 \sin y$

$f_1(x,y) = 2x \sin y$ and $f_2(x,y) = x^2 \cos y$

2. $z = x^3 y^2 + x^4 y + y^4$

$\frac{\partial z}{\partial x} = 3x^2 y^2 + 4x^3 y$ and $\frac{\partial z}{\partial y} = 2x^3 y + x^4 + 4y^3$

Example Evaluate the following partial derivatives.

1. If $f(x,y) = e^{xy} \cdot \cos(x+y)$ then find $f_1(0, \pi)$

2. $\frac{\partial}{\partial z} \left(\frac{2xy}{1+xz+yz} \right) \Big|_{(3,-1,0)} = ?$

Answers

$$1. f(x,y) = \underbrace{e^{xy}}_I \cdot \underbrace{\cos(x+y)}_II$$

$$f_1(x,y) = \underbrace{y \cdot e^{xy}}_{I'} \cdot \underbrace{\cos(x+y)}_II - \underbrace{\sin(x+y)}_{II'} \cdot \underbrace{e^{xy}}_I$$

$$f_1(0,\pi) = \pi \cdot e^{0 \cdot \pi} \cdot \cos(0+\pi) - \sin(0+\pi) \cdot e^{0 \cdot \pi}$$

$$= \pi \cdot \cos(\pi) - \sin(\pi) = -\pi$$

2. Here, we may think the question as follows:

If $w(x,y,z) = \frac{2xy}{1+xz+yz}$, then find $w_3(3,-1,0)$.

↳ third variable of the function w , which is z .

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z} [2xy \cdot (1+xz+yz)^{-1}] = -2xy \cdot (1+xz+yz)^{-2} \cdot (x+y)$$

$$w_3(x,y,z) = -\frac{2xy(x+y)}{(1+xz+yz)^2} \text{ and } w_3(3,-1,0) = -\frac{2 \cdot 3 \cdot (-1)(3-1)}{(1+3 \cdot 0-1 \cdot 0)^2} = 12$$

Exercises

1. $f(x,y) = xy + x^2$
 $f_1(x,y) = ?$ $f_2(x,y) = ?$

2. $f(x,y,z) = x^3 y^4 z^5$
 $f_1(x,y,z) = ?$ $f_2(x,y,z) = ?$ $f_3(x,y,z) = ?$

3. $g(x,y,z) = \frac{xz}{y+z}$
 $g_1(-1,1,3) = ?$ $g_2(-1,1,3) = ?$ $g_3(-1,1,3) = ?$

4. $z = \frac{1}{\sqrt{x^2+y^2}}$
 $\frac{\partial z}{\partial x} \Big|_{(-3,4)} = ?$ $\frac{\partial z}{\partial y} \Big|_{(-3,4)} = ?$

5. $g(x_1, x_2, x_3, x_4) = \frac{x_1 - x_2^2}{x_3 + x_4^2}$

$$\frac{\partial g}{\partial x_4} \Big|_{(3,1,-1,2)} = ?$$

* TANGENT PLANES & NORMAL LINES

↳ Normal vector to $z=f(x,y)$ at $(a,b,f(a,b))$ is:

$$\vec{n} = f_1(a,b)\mathbf{i} + f_2(a,b)\mathbf{j} - \mathbf{k}$$

↳ Equation of the tangent plane to $z=f(x,y)$ at $(a,b,f(a,b))$ is:

$$z = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)$$

↳ The Normal line to $z=f(x,y)$ at $(a,b,f(a,b))$ has direction vector:

$f_1(a,b)\mathbf{i} + f_2(a,b)\mathbf{j} - \mathbf{k}$, and so, has equation:

$$\frac{x-a}{f_1(a,b)} = \frac{y-b}{f_2(a,b)} = \frac{z-f(a,b)}{-1}$$

(Make suitable modification if either $f_1(a,b)=0$ or $f_2(a,b)=0$.)

Examples Find a normal vector and equations of tangent plane and normal line to the graphs of following functions:

1. $z = \sin(xy)$ at the point where $x = (\frac{\pi}{3})$ and $y = -1$
2. $f(x,y) = e^{xy}$ at $(2,0)$
3. $f(x,y) = \ln(x^2+y^2)$ at $(1,2)$

Answers

$$1. f(\frac{\pi}{3}, -1) = \sin(-\frac{\pi}{3}) = -\sin(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2} \quad \text{Point: } (\frac{\pi}{3}, -1, -\frac{\sqrt{3}}{2})$$

$$f(x,y) = \sin(xy)$$

$$f_1(x,y) = y \cdot \cos(xy)$$

$$f_2(x,y) = x \cdot \cos(xy)$$

$$f_1(\frac{\pi}{3}, -1) = -\cos(-\frac{\pi}{3}) = -\frac{1}{2}$$

$$f_2(\frac{\pi}{3}, -1) = \frac{\pi}{3} \cos(-\frac{\pi}{3}) = \frac{\pi}{6}$$

Normal Vector: $\vec{n} = \frac{-1}{2}i + \frac{\pi}{6}j - k$

Tangent Plane: $z = -\frac{\sqrt{3}}{2} - \frac{1}{2}(x - \frac{\pi}{3}) + \frac{\pi}{6}(y + 1)$

Normal Line: $\frac{x - \pi/3}{-1/2} = \frac{y + 1}{\pi/6} = \frac{z + \sqrt{3}/2}{-1}$

2. $f(2,0) = e^{2 \cdot 0} = 1$ Point: $(2, 0, 1)$

$f(x,y) = e^{xy}$

$f_1(x,y) = y \cdot e^{xy}$

$f_2(x,y) = x \cdot e^{xy}$

$f_1(2,0) = 0$

$f_2(2,0) = 2$

Normal Vector: $\vec{n} = 2j - k$

Tangent Plane: $z = 1 + 2 \cdot (y - 0)$

Normal Line: $\frac{y}{2} = \frac{z-1}{-1}, x=2$

$z = 1 + 2y$

3. $f(1,2) = \ln(1^2 + 2^2) = \ln 5$ Point: $(1, 2, \ln 5)$

$f(x,y) = \ln(x^2 + y^2)$

$f_1(x,y) = \frac{2x}{x^2 + y^2}$

$f_2(x,y) = \frac{2y}{x^2 + y^2}$

$f_1(1,2) = \frac{2}{1+4} = \frac{2}{5}$

$f_2(1,2) = \frac{2 \cdot 2}{1+4} = \frac{4}{5}$

Normal Vector: $\vec{n} = \frac{2}{5}i + \frac{4}{5}j - k$

Tangent Plane: $z = \ln 5 + \frac{2}{5}(x-1) + \frac{4}{5}(y-2)$

Normal Line: $\frac{x-1}{2/5} = \frac{y-2}{4/5} = \frac{z-\ln 5}{-1}$

Exercises Find the equations of the tangent plane and normal line to the graph of the given functions at specified points.

1. $f(x,y) = x^2 - y^2$ at $(-2, 1)$
2. $f(x,y) = \cos\left(\frac{x}{y}\right)$ at $(\pi, 4)$
3. $f(x,y) = y \cdot e^{-x^2}$ at $(0, 1)$
4. $f(x,y) = \frac{2xy}{x^2 + y^2}$ at $(0, 2)$
5. $f(x,y) = \sqrt{1 + x^3 y^2}$ at $(2, 1)$

* HIGHER ORDER DERIVATIVES

→ If $z = f(x,y)$, we can calculate four partial derivatives of second order, two **pure** and two **mixed** as follows:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{\partial z}{\partial x} \right] = f_{11}(x,y) = f_{xx}(x,y)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left[\frac{\partial z}{\partial y} \right] = f_{22}(x,y) = f_{yy}(x,y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial z}{\partial y} \right] = f_{21}(x,y) = f_{yx}(x,y)$$

→ order is REVERSED in this notation

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left[\frac{\partial z}{\partial x} \right] = f_{12}(x,y) = f_{xy}(x,y)$$

Similarly, if, for example, $w = f(x,y,z)$ then:

$$\frac{\partial^5 w}{\partial y \partial x \partial y^2 \partial z} = \frac{\partial}{\partial y} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial z} \right) = f_{32212}(x,y,z)$$

$\begin{matrix} 3 & 2 & 2 & 1 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ z & y & y & x & y \end{matrix}$

Examples

1. Find the four second partial derivatives of $f(x,y) = x^3 y^4$
2. If $f(x,y,z) = e^{x-2y+3z}$ then find f_{223} , f_{232} and f_{322}

Answers

$$1. \quad f_1(x,y) = 3x^2 y^4 \qquad f_2(x,y) = 4x^3 y^3$$

$$f_{11}(x,y) = 6xy^4 \qquad f_{21}(x,y) = 12x^2 y^3$$

$$f_{12}(x,y) = 12x^2 y^3 \qquad f_{22}(x,y) = 12x^3 y^2$$

$$2. \quad w = e^{x-2y+3z}$$

$$\rightarrow f_{223}(x,y,z) = \frac{\partial}{\partial z} \frac{\partial}{\partial y} \left[\frac{\partial w}{\partial y} \right] = \frac{\partial}{\partial z} \frac{\partial}{\partial y} (-2e^{x-2y+3z})$$

$$= \frac{\partial}{\partial z} (4e^{x-2y+3z}) = \underline{\underline{12e^{x-2y+3z}}}$$

$$\rightarrow f_{232}(x,y,z) = \frac{\partial}{\partial y} \frac{\partial}{\partial z} \left[\frac{\partial w}{\partial y} \right] = \frac{\partial}{\partial y} \frac{\partial}{\partial z} (-2e^{x-2y+3z})$$

$$= \frac{\partial}{\partial y} (-6e^{x-2y+3z}) = \underline{\underline{12e^{x-2y+3z}}}$$

$$\rightarrow f_{322}(x,y,z) = \frac{\partial}{\partial y} \frac{\partial}{\partial y} \left[\frac{\partial w}{\partial z} \right]$$

$$= \frac{\partial}{\partial y} \frac{\partial}{\partial y} (3e^{x-2y+3z}) = \frac{\partial}{\partial y} (-6e^{x-2y+3z}) = \underline{\underline{12e^{x-2y+3z}}}$$

! Always same result for continuous functions, if variables are repeated the same number of times!

Exercises

1. Find all the second partial derivatives of $f(x,y) = \ln[1 + \sin(xy)]$
2. If $w = e^{3x+4y} \cdot \sin(5z)$ then find $f_{zyyxy}(x,y,z)$

* CHAIN RULE

Let $z = f(x, y)$ and $x = u(t)$; $y = v(t)$. Then:

$$g(t) = z = f[u(t), v(t)]$$

$$g'(t) = \frac{dz}{dt} = f_1[u(t), v(t)] \cdot u'(t) + f_2[u(t), v(t)] \cdot v'(t)$$

$$= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Now, consider a function z of two variables: x and y , each of which is in turn a function of two others: s and t .

Then: $g(s, t) = z = f[u(s, t), v(s, t)]$

$$g_1(s, t) = f_1[u(s, t), v(s, t)] \cdot u_1(s, t) + f_2[u(s, t), v(s, t)] \cdot v_1(s, t)$$

$$\text{or } \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\text{and } g_2(s, t) = f_1[u(s, t), v(s, t)] \cdot u_2(s, t) + f_2[u(s, t), v(s, t)] \cdot v_2(s, t)$$

$$\text{or } \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Matrix form of the latter is:

$$\begin{bmatrix} \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

Examples 4

1) $z = \sin(x^2y)$ where $x = st^2$ and $y = s^2 + \frac{1}{t}$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Ans 1

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} = 2xy \cos(x^2y) t^2 + x^2 \cos(x^2y) \cdot 2s \\ &= \left[2st^2 \left(s^2 + \frac{1}{t} \right) t^2 + s^2 t^4 \cdot 2s \right] \cos \left[s^2 t^4 \left(s^2 + \frac{1}{t} \right) \right] \\ &= (4s^3 t^4 + 2st^3) \cos(s^4 t^4 + s^2 t^3) \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} = 2xy \cos(x^2y) 2st + x^2 \cos(x^2y) \frac{(-1)}{t^2} \\ &= \left[2st^2 \left(s^2 + \frac{1}{t} \right) \cdot 2st - \frac{s^2 t^4}{t^2} \right] \cdot \cos \left[s^2 t^4 \left(s^2 + \frac{1}{t} \right) \right] \\ &= (4s^4 t^3 + 3s^2 t^2) \cos(s^4 t^4 + s^2 t^3) \end{aligned}$$

2) Find $\frac{\partial}{\partial x} f(x^2y, x+2y)$ and $\frac{\partial}{\partial y} f(x^2y, x+2y)$ in terms of the partial derivatives of f

Ans 2

$$\begin{aligned} \frac{\partial}{\partial x} f(x^2y, x+2y) &= f_1(x^2y, x+2y) \frac{\partial}{\partial x} (x^2y) + f_2(x^2y, x+2y) \cdot \frac{\partial}{\partial x} (x+2y) \\ &= 2xy f_1(x^2y, x+2y) + f_2(x^2y, x+2y) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial y} f(x^2y, x+2y) &= f_1(x^2y, x+2y) \frac{\partial}{\partial y} (x^2y) + f_2(x^2y, x+2y) \frac{\partial}{\partial y} (x+2y) \\ &= x^2 f_1(x^2y, x+2y) + 2f_2(x^2y, x+2y) \end{aligned}$$

3) Express the partial derivatives of $z = h(s, t) = f[g(s, t)]$ in terms of partial derivatives of f and g .

Ans 3

$$h_1(s, t) = \frac{\partial z}{\partial s} = f'[g(s, t)] \cdot g_1(s, t)$$

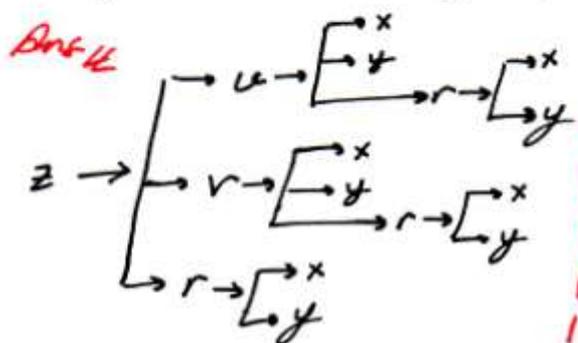
$$h_2(s, t) = \frac{\partial z}{\partial t} = f'[g(s, t)] \cdot g_2(s, t)$$

4) Find $\frac{dz}{dt}$ where $z = f(x, y, t)$, $x = g(t)$ and $y = h(t)$

Ans 4
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial z}{\partial t}$$

$$= f_1(x, y, t) \cdot g'(t) + f_2(x, y, t) \cdot h'(t) + f_3(x, y, t)$$

5) Write the appropriate chain rule for $\frac{\partial z}{\partial x}$ where $z = f(u, v, r)$, $u = g(x, y, r)$, $v = h(x, y, r)$ and $r = k(x, y)$



Then,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial u} \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial z}{\partial r} \frac{\partial r}{\partial x}$$

Exercises 4

1. $\frac{\partial w}{\partial t}$ where $w = f(x, y, z)$ and $x = g(s, t)$, $y = h(s, t)$, $z = k(s, t)$

2. $\frac{dw}{dx}$ where $w = f(x, y, z)$ and $y = g(x, z)$, $z = h(x)$

3. $\frac{dw}{dt}$ where $w = f(x, y)$ and $x = g(r, s)$, $y = h(r, t)$, $r = k(s, t)$, $s = m(t)$

4. $\frac{\partial z}{\partial x}$ where $z = \tan^{-1}\left(\frac{u}{v}\right)$ and $u = 2x + y$, $v = 3x - y$

5. $\frac{dz}{dt}$ where $z = txy^2$ and $x = t + \ln(y + t^2)$, $y = e^t$

6. $\frac{\partial}{\partial x} f(y^2, x^2)$

7. $\frac{\partial}{\partial x} f[f(x, y), f(x, y)]$

* LINEAR APPROXIMATIONS & DIFFERENTIALS

The tangent line to the graph $y=f(x)$ at $x=a$ provides a convenient approximation for values of $f(x)$ near a :

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

Here, $L(x)$ is the linearization of f at a since at point $x=a$, graph of $f(x)$ is the tangent line.

↳ The linearization (linear approximation) of $f(x,y)$ at point (a,b) is given by:

$$f(x,y) \approx L(x,y) = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)$$

Example 4 Find an approximate value for $f(x,y) = \sqrt{2x^2 + e^{2y}}$ at $(2,2; -0,2)$

Ans 4 $f(2,0) = \sqrt{2 \cdot 2^2 + e^{2 \cdot 0}} = \sqrt{8+1} = 3$

$$f_1(x,y) = \frac{4x}{2\sqrt{2x^2 + e^{2y}}} \quad f_2(x,y) = \frac{2 \cdot e^{2y}}{2\sqrt{2x^2 + e^{2y}}}$$

$$f_1(2,0) = \frac{4 \cdot 2}{2\sqrt{2 \cdot 2^2 + e^{2 \cdot 0}}} = \frac{4}{3} \quad f_2(2,0) = \frac{2 \cdot e^{2 \cdot 0}}{2\sqrt{2 \cdot 2^2 + e^{2 \cdot 0}}} = \frac{1}{3}$$

Thus, $L(x,y) = 3 + \frac{4}{3}(x-2) + \frac{1}{3}(y-0)$ and

$$f(2,2; -0,2) \approx L(2,2; -0,2) = 3 + \frac{4}{3}(2,2-2) + \frac{1}{3}(-0,2-0) = 3,2$$

Exercises 4 Use suitable linearization to find approximate values for the given functions at the indicated points.

- $f(x,y) = x^2 y^3$ at $(3,1; 0,9)$
- $f(x,y) = \sin(\pi xy + \ln y)$ at $(0,01; 1,05)$
- $f(x,y,z) = \sqrt{x+2y+3z}$ at $(1,9; 1,8; 1,1)$

* Differentials

Let $z = f(x_1, x_2, \dots, x_n)$ be given. We may construct a differential dz or df of the function at a point:

$$dz = df = \frac{\partial z}{\partial x_1} dx_1 + \frac{\partial z}{\partial x_2} dx_2 + \dots + \frac{\partial z}{\partial x_n} dx_n$$

$$= f_1(x_1, \dots, x_n) dx_1 + \dots + f_n(x_1, \dots, x_n) dx_n$$

For a differentiable function f , the differential df is an approximation to the change Δf in value of the function given by:

$$\Delta f = f(x_1 + d_1, x_2 + d_2, \dots, x_n + d_n) - f(x_1, x_2, \dots, x_n)$$

The error in this approximation is small compared to the distance between the two points in the domain of f :

$$\frac{\Delta f - df}{\sqrt{dx_1^2 + dx_2^2 + \dots + dx_n^2}} \rightarrow 0 \text{ if all } dx_i \rightarrow 0 \quad 1 \leq i \leq n$$

Example Find the percentage change in the period $T = 2\pi \sqrt{\frac{L}{g}}$ of a small pendulum if the length L of the pendulum increases by 2% and the acceleration of gravity g decreases by 0.1%.

Ans $dT = \frac{\partial T}{\partial L} dL + \frac{\partial T}{\partial g} dg = \frac{2\pi}{2\sqrt{Lg}} dL - \frac{2\pi \sqrt{L}}{2g^{3/2}} dg$

We are given that $dL = \frac{2}{100} L$ and $dg = -\frac{0.1}{100} g$.

Thus, $dT = \frac{2}{100} \cdot 2\pi \sqrt{\frac{L}{g}} - \left(-\frac{0.1}{100}\right) \frac{2\pi}{2} \sqrt{\frac{L}{g}} = \frac{13}{1000} T$.

Therefore, the period T of pendulum increases by 1.3%.

Exercise The edges of a rectangular box are measured to within an accuracy of 1% of their values. What is the approximate maximum percentage error in

- a) the calculated volume of the box?
- b) the calculated area of the faces of the box?
- c) the calculated length of a diagonal of the box?