LECTURE NOTES CHAPTER 8
SIMULATION

Random Variate Generation (next:)

Having obtained random numbers that are Uniform(0;1) and independent, we'll learn transforming them to other Random Variables that have specific distributions:

(I) INVERSE TRANSFORM TECHNIQUE

Step 1. Compute the cdf of the desired R.V. X: $\boxed{F(X)}$

Step 2. Set $\boxed{F(X)=R}$ on the range of X

Step 3. Solve the equation $F(X)=R$ in terms of X: $\boxed{X=F^{-1}(R)}$

Step 4. Given $R_1, R_2, \ldots, R_N$: Obtain $\boxed{X_i=F^{-1}(R_i)}$

CONTINUOUS DISTRIBUTIONS:

Exponential Distribution

$X \sim \text{Exponential}(\lambda)$

$$f(x)=\begin{cases}\lambda . e^{-\lambda x} & x \geq 0 \\ 0 & o.w\end{cases} \qquad E(X)=\frac{1}{\lambda} \qquad Var(X)=\frac{1}{\lambda^2}$$

$$F(x)=\int_{-\infty}^{x} f(w)dw=\int_{0}^{x} f(w)dw=\int_{0}^{x}(\lambda . e^{-\lambda w})dw$$

$$=\lambda . \frac{-1}{\lambda} . e^{-\lambda w}\Big]_0^x=-(e^{-\lambda x}-1)=1-e^{-\lambda x} \quad x \geq 0$$

$$\boxed{F(x)=\begin{cases}1-e^{-\lambda x} & x \geq 0 \\ 0 & x<0\end{cases}}$$

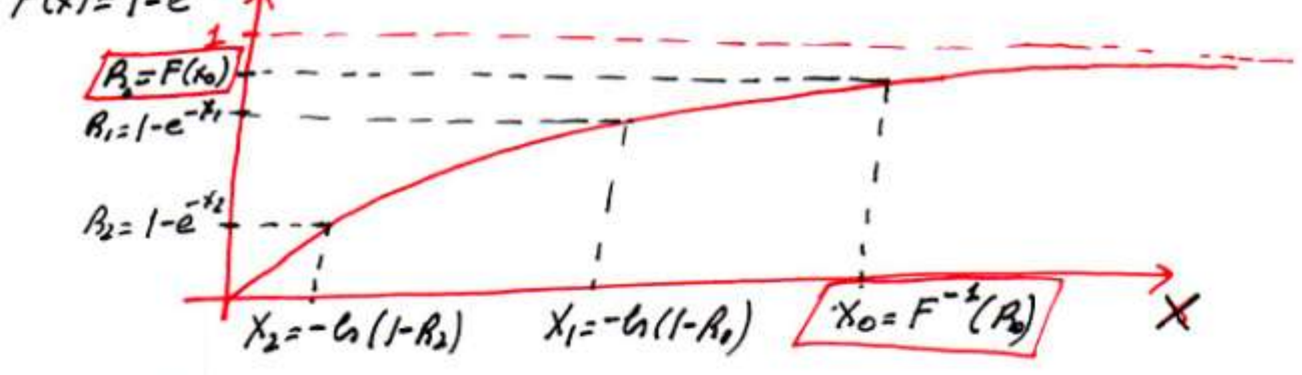
$$F(X) = R$$
$$1 - e^{-\lambda X} = R$$
$$1 - R = e^{-\lambda X}$$
$$\ln(1-R) = -\lambda X$$
$$\boxed{X = F^{-1}(R) = -\frac{1}{\lambda} \cdot \ln(1-R)}$$

Note that, $1-R \sim$ Uniform $(0;1)$ because $R \sim$ Uniform$(0;1)$

So, we can say in short;

$$\boxed{X = F^{-1}(R) = -\frac{1}{\lambda} \cdot \ln R}$$

Examine the logic of Inverse transform technique by the help of following graph; for $\lambda = 1$



17. Lead times have been found to be exponentially distributed with mean 3.7 days. Generate five random lead times from this distribution.

17) Mean $= 3,7 = \frac{1}{\lambda} \Rightarrow X_i = -3,7 \cdot \ln R_i$

let the Random numbers are Given (or from the table):

i	1	2	3	4	5
R_i	0,2031	0,8211	0,0390	0,6207	0,8520
X_i	$-3,7 \cdot \ln 0,2031$ $= 5,898$	$-3,7 \cdot \ln 0,8211$ $= 0,725$	$-3,7 \cdot \ln 0,0390$ $= 12,004$	$-3,7 \cdot \ln 0,6207$ $= 1,765$	$-3,7 \cdot \ln 0,8520$ $= 0,593$

1. Develop a random-variate generator for a random variable X with the pdf

$$f(x) = \begin{cases} e^{2x}, & -\infty < x \le 0 \\ e^{-2x}, & 0 < x < \infty \end{cases}$$

1) $F(x) = \int_{-\infty}^{x} f(w)\,dw$

for $-\infty < x \le 0$: $F(x) = \int_{-\infty}^{x} e^{2w}\,dw = \frac{e^{2w}}{2}\Big]_{-\infty}^{x} = \frac{1}{2}\left(e^{2x} - e^{-\infty}\right) = \frac{e^{2x}}{2}$

$$F(0) = \frac{e^{0}}{2} = \frac{1}{2}$$

for $0 < x < \infty$: $F(x) = \frac{1}{2} + \int_{0}^{x} e^{-2w} = \frac{1}{2} - \frac{1}{2} \cdot e^{-2w}\Big]_{0}^{x} = \frac{1}{2} - \frac{1}{2}\left[e^{-2x} - 1\right]$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} e^{-2x} = 1 - \frac{1}{2} \cdot e^{-2x}$$

$$F(x) = \begin{cases} \frac{1}{2} e^{2x} & -\infty < x \le 0 \\ 1 - \frac{1}{2} e^{-2x} & 0 < x < \infty \end{cases}$$

Since 0 is a break-point and $F(0) = \frac{1}{2}$; we have:

$0 \le R \le \frac{1}{2}$: $F(X) = R$

$$\frac{1}{2} e^{2x} = R$$

$$e^{2x} = 2R$$

$$X = \frac{1}{2} \ln(2R)$$

$\frac{1}{2} < R \le 1$: $F(X) = R$

$$1 - \frac{1}{2} e^{-2x} = R$$

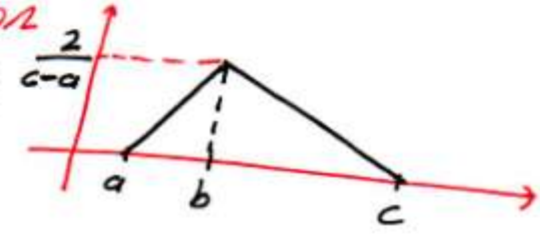
$$2.(1 - R) = e^{-2x}$$

$$X = -\frac{1}{2} \ln(2.(1-R))$$

$$X = \begin{cases} \frac{1}{2} \ln(2R) & 0 \le R \le 0{,}5 \\ -\frac{1}{2} \ln[2(1-R)] & 0{,}5 < R < 1 \end{cases}$$

Triangular Distribution

$X \sim \text{Trianguler}(a;b;c)$



$$f(x)=\begin{cases}\dfrac{2(x-a)}{(b-a)(c-a)} & a\le x<b\\[2ex] \dfrac{2(c-x)}{(c-b)(c-a)} & b<x\le c\\[2ex] 0 & \text{o.w.}\end{cases}$$

$$E(X)=\frac{a+b+c}{3}$$

$$\text{Mode}=b=3E(X)-(a-c)$$

Note that triangular distribution pdf is usual line equation. Height $\frac{2}{c-a}$ make total Area $=1$.

Cdf. $F(x)$ is given by following formula but we can also calculate it as in last exercise in a usual manner.

$$F(x)=\begin{cases}0 & x\le a\\[1ex] \dfrac{(x-a)^2}{(b-a)(c-a)} & a<x\le b\\[2ex] 1-\dfrac{(c-x)^2}{(c-b)(c-a)} & b<x\le c\\[2ex] 1 & x>c\end{cases}$$

2. Develop a generation scheme for the triangular distribution with pdf

$$f(x)=\begin{cases}\frac{1}{2}(x-2), & 2\le x\le 3\\ \frac{1}{2}\left(2-\frac{x}{3}\right), & 3<x\le 6\\ 0, & \text{otherwise}\end{cases}$$

Generate 10 values of the random variate, compute the sample mean, and compare it to the true mean of the distribution.

3. Develop a generator for a triangular distribution with range (1, 10) and mode at $x=4$.
4. Develop a generator for a triangular distribution with range (1, 10) and a mean of 4.

2) I'll NOT use formula of F(x), but drive F(x) manually :)

$$2 \le x \le 3: \quad F(x) = \int_2^x \frac{1}{2}(w-2)\,dw = \frac{1}{2}\left(\frac{w^2}{2} - 2w\right]_2^x = \frac{1}{2}\left[\frac{x^2}{2} - 2x - \left(\frac{4}{2} - 4\right)\right]$$

$$= \frac{1}{2}\left(\frac{x^2 - 4x + 4}{2}\right) = \frac{(x-2)^2}{4} \quad ; \quad F(3) = \frac{(3-2)^2}{4} = \frac{1}{4}$$

$$3 < x \le 6: \quad F(x) = \frac{1}{4} + \int_3^x \frac{1}{2}\left(2 - \frac{w}{3}\right)dw = \frac{1}{4} + \frac{1}{2}\left(2w - \frac{w^2}{6}\right]_3^x$$

$$= \frac{1}{4} + \frac{1}{2}\left[2x - \frac{x^2}{6} - \left(6 - \frac{\cancel{9}^{\,3}}{\cancel{6}_{\,2}}\right)\right] = \frac{1}{4} + \frac{1}{2}\left[\frac{12x - x^2 - 27}{6}\right]$$

$$= \frac{1}{4} + \frac{1}{2}\left(-\frac{(x^2 - 12x + 36) + 9}{6}\right) = \frac{1}{4} + \frac{1}{2}\left(\frac{3}{2} - \frac{(x-6)^2}{6}\right) = 1 - \frac{(x-6)^2}{12}$$

$$F(x) = \begin{cases} 0 & x < 2 \\ \frac{(x-2)^2}{4} & 2 \le x \le 3 \; ; \; F(3) = \frac{1}{4} \\ 1 - \frac{(x-6)^2}{12} & 3 < x < 6 \\ 1 & x > 6 \end{cases}$$

$0 \le R \le \frac{1}{4}$: $F(X) = R$

$$\frac{(X-2)^2}{4} = R$$

$$(X-2)^2 = 4R$$

$$X = 2 + \sqrt{4R}$$

$\frac{1}{4} \le R \le 1$: $F(X) = R$

$$1 - \frac{(x-6)^2}{12} = R$$

$$12(1-R) = (6-X)^2$$

$= (x-6)^2$ but X should be within the range

$$X = 6 - \sqrt{12(1-R)}$$

$$X = \begin{cases} 2 + \sqrt{4R} & 0 \le R \le 0{,}25 \\ 6 - \sqrt{12(1-R)} & 0{,}25 < R \le 1 \end{cases}$$

3) $a=1;\ b=4;\ c=10$ $\quad (b-a)(c-a)=3\cdot 9=27$ and
$(c-b)(c-a)=6\cdot 9=54$

$$F(x)=\begin{cases} 0 & x\le 1 \\ \dfrac{(x-1)^2}{27} & 1<x\le 4 \\ 1-\dfrac{(10-x)^2}{54} & 4<x\le 10 \\ 1 & x>10 \end{cases} \qquad F(4)=\frac{(4-1)^2}{27}=\frac{1}{3}$$

$0\le R\le \frac{1}{3}$: $F(x)=R$

$$\frac{(x-1)^2}{27}=R$$

$$x=1+\sqrt{27R}$$

$\frac{1}{3}\le R\le 1$: $F(x)=R$

$$1-\frac{(10-x)^2}{54}=R$$

$$54(1-R)=(10-x)^2$$

$$x=10-\sqrt{54(1-R)}$$

$$X=\begin{cases} 1+\sqrt{27R} & 0\le R\le 0{,}3333 \\ 10-\sqrt{54(1-R)} & 0{,}3333<R\le 1 \end{cases}$$

4) Mode $=b=3\cdot 4-(10-1)=12-9=3$

Apply the same precedure in Q.3 with $a=1;\ b=3;\ c=10$

Empirical Continuous Distributions

If we have no theoretical distribution that provides a good model for the input data for a continuous Random Variable, we Rank the observations, plot the points of Empirical cdf $S_N(x)$ and combine the plotted points to approximate $F(x)$ linearly. Next are two examples:

Example: Five observations of fire-crew response times to incoming alarms have been collected to be used in a simulation investigating possible alternative staffing and crew-scheduling policies. The data are:

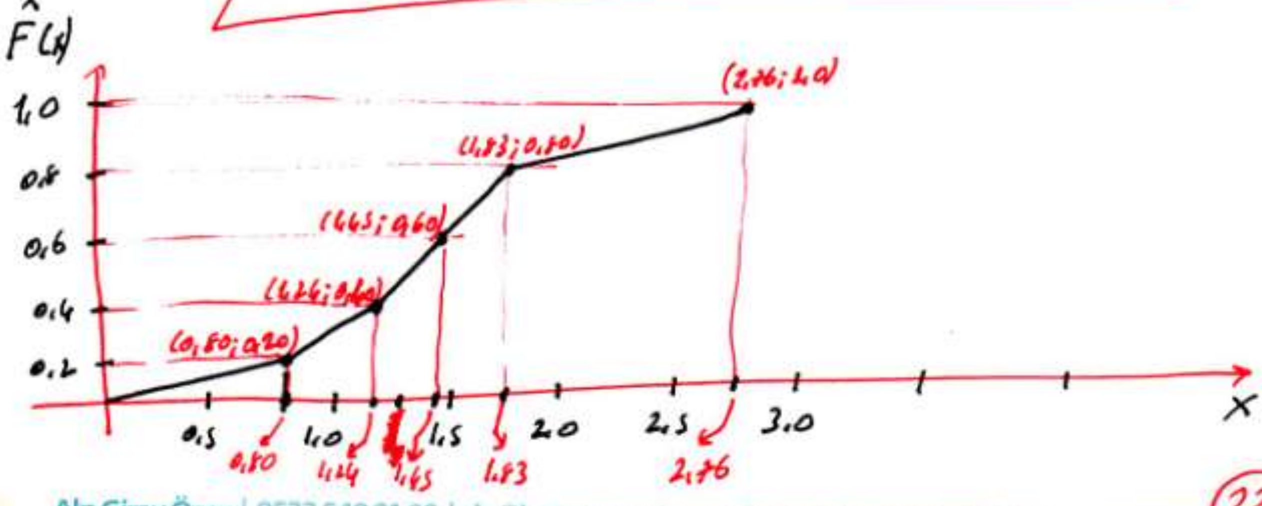
2,76 1,83 0,80 1,45 1,24

Answer:

i	Interval $x_{(i-1)} < x \leq x_{(i)}$	Probability $1/n$	$\hat{F}$: Cum. Prob. i/n	Slope $a_i = \frac{x_i - x_{(i-1)}}{1/n}$
1	$0{,}0 < x \leq 0{,}80$	0,2	0,2	$\frac{0{,}80 - 0{,}00}{0{,}2} = 4{,}00$
2	$0{,}80 < x \leq 1{,}24$	0,2	0,4	$\frac{1{,}24 - 0{,}80}{0{,}2} = 2{,}20$
3	$1{,}24 < x \leq 1{,}45$	0,2	0,6	$\frac{1{,}45 - 1{,}24}{0{,}2} = 1{,}05$
4	$1{,}45 < x \leq 1{,}83$	0,2	0,8	$\frac{1{,}83 - 1{,}45}{0{,}2} = 1{,}90$
5	$1{,}83 < x \leq 2{,}76$	0,2	1,0	$\frac{2{,}76 - 1{,}83}{0{,}2} = 4{,}65$

Note that a_i is the slope of the i^{th} interval's straight line

$$\boxed{X = \hat{F}^{-1}(R) = x_{(i-1)} + a_i \cdot \left(R - \frac{(i-1)}{n} \right)}$$



So, the linear equations for i^{th} interval, namely the Empirical Random Variate generator is as follows;

$$X = \begin{cases} = 4R & 0 < R \leq 0{,}20 \\ = 0{,}80 + 2{,}20 \cdot (R - 0{,}2) & 0{,}20 < R \leq 0{,}40 \\ = 1{,}24 + 1{,}05 \cdot (R - 0{,}4) & 0{,}40 < R \leq 0{,}60 \\ = 1{,}45 + 1{,}90 \cdot (R - 0{,}6) & 0{,}60 < R \leq 0{,}80 \\ = 1{,}83 + 4{,}65 \cdot (R - 0{,}8) & 0{,}80 < R \leq 1{,}00 \end{cases}$$

11. Data have been collected on service times at a drive-in bank window at the Shady Lane National Bank. This data are summarized into intervals as follows:

a_i	Interval (Seconds)	Frequency	Rel Freq.	Cum. Freq.	Slope a_i
$15 \div 0{,}07$	15–30	10	0,07	0,07	214,3
$15 \div 0{,}13$	30–45	20	0,13	0,20	115,4
$15 \div 0{,}17$	45–60	25	0,17	0,37	88,2
$30 \div 0{,}23$	60–90	35	0,23	0,60	130,4
$30 \div 0{,}20$	90–120	30	0,20	0,80	150,0
$60 \div 0{,}13$	120–180	20	0,13	0,93	461,5
$120 \div 0{,}07$	180–300	10	0,07	1,00	1714,3
		+ 150			

Set up a table like Table 8.3 for generating service times by the table-lookup method, and generate five values of service time, using four-digit random numbers.

$$X = \begin{cases} 15 + 214{,}3 \cdot R & 0 < R \leq 0{,}07 \\ 30 + 115{,}4\,(R - 0{,}07) & 0{,}07 < R \leq 0{,}20 \\ 45 + 88{,}2 \cdot (R - 0{,}20) & 0{,}20 < R \leq 0{,}37 \\ 60 + 130{,}4 \cdot (R - 0{,}37) & 0{,}37 < R \leq 0{,}60 \\ 90 + 150{,}0 \cdot (R - 0{,}60) & 0{,}60 < R \leq 0{,}80 \\ 120 + 461{,}5 \cdot (R - 0{,}80) & 0{,}80 < R \leq 0{,}93 \\ 180 + 1714{,}3 \cdot (R - 0{,}93) & 0{,}93 < R \leq 1{,}00 \end{cases}$$

10. Times to failure for an automated production process have been found to be randomly distributed with a Weibull distribution with parameters $\beta = 2$ and $\alpha = 10$. Derive Equation (8.6), and then use it to generate five values from this Weibull distribution, using five random numbers taken from Table A.1.

10) Weibull Distribution

$$X \sim \text{Weibull}(\alpha;\beta)$$

$$f(x) = \begin{cases} \frac{\beta}{\alpha^\beta} x^{\beta-1} e^{-(x/\alpha)^\beta} & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

$$F(x) = \int_0^x f(w)\,dw = \int_0^x \frac{\beta}{\alpha^\beta} \cdot w^{\beta-1} \cdot e^{-\left(\frac{w}{\alpha}\right)^\beta} dw$$

$(w/\alpha)^\beta = u$

$(\beta/\alpha) \cdot (w/\alpha)^{\beta-1} dw = \frac{\beta}{\alpha^\beta} \cdot w^{\beta-1} dw = du$

$$F(x) = \int_0^{\left(\frac{x}{\alpha}\right)^\beta} e^{-u}\,du = -e^{-u}\Big]_0^{\left(\frac{x}{\alpha}\right)^\beta} = 1 - e^{-(x/\alpha)^\beta} \quad x \geq 0$$

$$F(X) = R$$

$$1 - e^{-(x/\alpha)^\beta} = R$$

$$1 - R = e^{-(x/\alpha)^\beta}$$

$$\ln(1-R) = -(x/\alpha)^\beta$$

$$[-\ln(1-R)]^{1/\beta} = +\frac{x}{\alpha}$$

$$\boxed{X = +\alpha \cdot [-\ln R]^{1/\beta}}$$

Since R and $1-R$ has same distribution

$\beta = 2;\ \alpha = 10 \Rightarrow X_i = +10 \cdot [-\ln R_i]^{1/2}$

i	1	2	3	4	5
R_i	0,8252	0,7926	0,1257	0,1093	0,4756
X_i	4,383	4,821	14,401	14,878	8,621

$10[-\ln 0{,}8252]^{1/2}$ — — — —

19. A machine is taken out of production either if it fails or after 5 hours, whichever comes first. By running similar machines until failure, it has been found that time to failure, X, has the Weibull distribution with $\alpha = 8$, $\beta = 0.75$, and $\nu = 0$ (refer to Sections 5.4 and 8.1.3). Thus, the time until the machine is taken out of production can be represented as $Y = \min(X, 5)$. Develop a step-by-step procedure for generating Y.

19) Step 0 $i=1$, set N

Step 1 Generate R_i

Step 2 $X_i = 8 \cdot [-\ln R_i]^{0,75}$

Step 3 If $X_i > 5$ then $y_i = X_i$

o.w. $y_i = 5$

Step 4 If $N = i$, stop.

o.w $i \leftarrow i+1$ and GOTO step 1

Standard Normal Distribution

Schmeiser Method

$$Z = \frac{R^{0,135} - (1-R)^{0,135}}{0,1975}$$

Box and Muller Method

$$Z_1 = (-2 \ln R_1)^{1/2} \cos(2\pi R_2)$$
$$Z_2 = (-2 \ln R_1)^{1/2} \sin(2\pi R_2)$$

Note That $X_i = \mu + Z_i \cdot \sigma \sim \text{Normal}(\mu; \sigma^2)$

18. Regular maintenance of a production routine has been found to vary and has been modeled as a normally distributed random variable with mean 33 minutes and variance 4 minutes^2. Generate five random maintenance times from the given distribution.

18) Schmeiser

i	1	2	3	4	5
R_i	0,3096	0,7776	0,0292	0,0152	0,2634
Z_i	-0,494	0,761	-1,901	-2,176	-0,630
$33+2Z_i = X_i$	32,01	34,52	29,20	28,65	31,74

Continuous Uniform Distribution

* Remember, $X \sim$ Uniform $(a; b)$

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & o.w \end{cases} \qquad F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > a \end{cases}$$

Then; $F(X) = R$

$$\frac{X-a}{b-a} = R$$

$\boxed{X = a + (b-a).R}$ generates Uniform Random Numbers over a and b from R

Example: Generate 5 Random Numbers $X \sim$ Uniform$[-11, 17]$

Answer:

$$F(x) = \begin{cases} 0 & x < -11 \\ \frac{x+11}{28} & -11 \le x \le 17 \\ 1 & x > 17 \end{cases}$$

$$F(X) = R$$

$$\frac{X+11}{28} = R \Rightarrow \underline{X = 28R - 11}$$

i	1	2	3	4	5
R_i	0,3045	0,0474	0,4683	0,5122	0,2620
X_i	−2,47	−9,67	1,55	3,34	−3,66

$= 28 . 0,3045 - 11$ — — —

DISCRETE DISTRIBUTIONS:

Empirical Discrete Distribution;

Example 4: At the end of any day, the # of shipments on a loading dock company is either 0, 1 or 2 with observed relative frequency of occurance of 0,50; 0,30; 0,20 respectively. Simulate 10 shipments and estimate the mean shipment/day. Compare it with the real mean.

Answer:

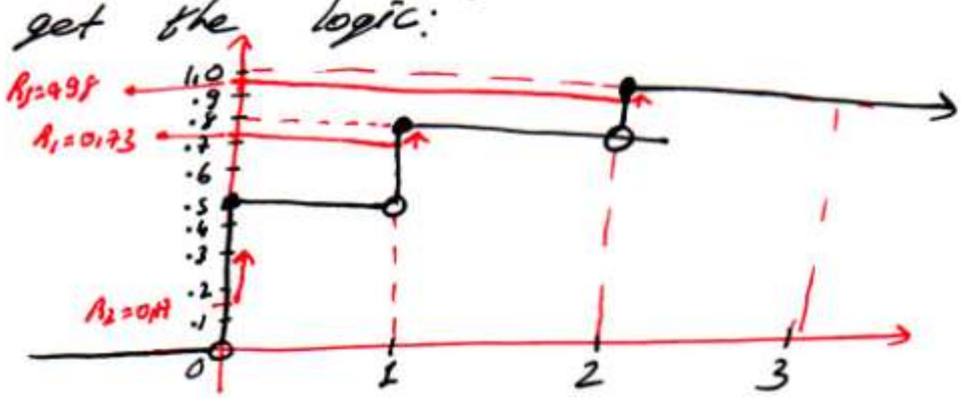
x	$p(x)$	$F(x)$
0	0,50	0,50
1	0,30	0,80
2	0,20	1.00

$$F(x)=\begin{cases} 0 & x<0 \\ 0,5 & 0\le x<1 \\ 0,8 & 1\le x<2 \\ 1,0 & x\ge 2 \end{cases}$$

Then, $F(x)=R \Rightarrow X=F^{-1}(R)$ gives;

$$X=\begin{cases} 0 & R\le 0,5 \\ 1 & 0,5<R\le 0,8 \\ 2 & 0,8<R\le 1,0 \end{cases}$$

Observe the following graph to get the logic:



i	R_i	X_i
1	0,73	1
2	0,17	0
3	0,98	2
4	0,30	0
5	0,04	0
6	0,44	0
7	0,51	1
8	0,26	0
9	0,75	1
10	0,97	2
		$\Sigma X_i = 7$

$$\hat{\mu}=\bar{X}=\frac{\Sigma X_i}{n}=\frac{7}{10}=\underline{0,70}$$

$$\mu=E(X)=0,50.0+0,30.1+0,20.2=\underline{0,50}$$

Discrete Uniform Distribution

Remember; $X \sim$ Discrete Uniform (k)

$p(x) = \frac{1}{k}, \quad x = 1, 2, \dots, k$

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{k} & 1 \le x < 2 \\ \frac{2}{k} & 2 \le x < 3 \\ \vdots & \\ \frac{k-1}{k} & k-1 \le x < k \\ 1 & k \le x \end{cases}$$

Then; $F(X) = R$ implies

$F(x_{i-1}) = r_{i-1} < R \le r_i = F(x_i)$

$r_{i-1} = \frac{i-1}{k} < R \le r_i = \frac{i}{k}$

$i - 1 < Rk \le i$

$Rk \le i < Rk + 1 \Rightarrow \boxed{X = \lceil Rk \rceil}$ → Round UP function!

13. For a preliminary version of a simulation model, the number of pallets, X, to be loaded onto a truck at a loading dock was assumed to be uniformly distributed between 8 and 24. Devise a method for generating X, assuming that the loads on successive trucks are independent. Use the technique of Example 8.5 for discrete uniform distributions. Finally, generate loads for 10 successive trucks by using four-digit random numbers.

13) Note that, $Y \sim$ Discrete Uniform (17) → $24 - 8 + 1 = 17$ Numbers

Then, $X = Y + 7$ → $8 - 1 = 7$ because $Y = 1, 2, \dots, 17$
$X = 8, 9, \dots, 24$

Step 0 Set $i = 1$
Step 1 Generate R_i
Step 2 $X_i = \lceil 17 \cdot R_i \rceil$
Step 3 $Y_i = X_i + 7$
Step 4 If $N = i$ stop.
o.w. $i \leftarrow i + 1$ and GOTO Step 1

i	1	2	3	4	5
R_i	0,8832	0,7686	0,9691	0,3026	0,8670
X_i	16	13	17	6	15
Y_i	23	20	24	13	22

$\lceil 17 \cdot 0,8832 \rceil$ → 16; $16 + 7$ → 23

i	6	7	8	9	10
R_i	0,0532	0,2143	0,1111	0,5566	0,6113
X_i	1	4	2	10	11
Y_i	8	11	9	17	18

9. The cdf of a discrete random variable X is given by

$$F(x) = \frac{x(x+1)(2x+1)}{n(n+1)(2n+1)}, \quad x = 1, 2, \ldots, n$$

When $n = 4$, generate three values of X, using $R_1 = 0.83$, $R_2 = 0.24$, and $R_3 = 0.57$.

9) $n=4 \Rightarrow F(x) = \dfrac{x(x+1)(2x+1)}{4.5.9} \quad x = 1,2,3,4$

$F(1) = \dfrac{1.2.3}{4.5.9} = 0{,}033 \quad F(2) = \dfrac{2.3.5}{4.5.9} = 0{,}167 \quad F(3) = \dfrac{3.6.7}{4.5.9} = 0{,}667 \quad F(4) = 1$

$$F(x) = \begin{cases} 0 & x < 1 \\ 0{,}033 & 1 \le x < 2 \\ 0{,}167 & 2 \le x < 3 \\ 0{,}667 & 3 \le x < 4 \\ 1 & x \geq 4 \end{cases} \Rightarrow X = \begin{cases} 1 & R \le 0{,}033 \\ 2 & 0{,}033 < R \le 0{,}167 \\ 3 & 0{,}167 < R \le 0{,}667 \\ 4 & R > 0{,}667 \end{cases}$$

i	1	2	3
R_i	0,83	0,24	0,57
X_i	4	3	4

Geometric Distribution

$X \sim \text{Geometric}(p) \quad E(X) = \dfrac{1}{p}$

$p(x) = p(1-p)^x, \quad x = 0, 1, 2, \ldots$

(Usually x starts from 1 but we have started it from 0 for ease of calculations)

$$F(x) = \sum_{w=0}^{x} p(1-p)^w = p \cdot \sum_{w=0}^{x} (1-p)^w = \underbrace{p \cdot \frac{1-(1-p)^{x+1}}{1-(1-p)}}_{\text{Finite sum formula}} = 1-(1-p)^{x+1}$$

$$\boxed{F(x) = 1 - (1-p)^{x+1}}$$

$$F(x-1) = 1-(1-p)^x < R \le 1-(1-p)^{x+1} = F(x)$$

$$(1-p)^{x+1} \le 1-R < (1-p)^x$$

$$(x+1)\ln(1-p) \le \ln(1-R) < x.\ln(1-p)$$

$$x+1 \geq \frac{\ln(1-R)}{\ln(1-p)_{<0}} \qquad x > \frac{\ln(1-R)}{\ln(1-p)_{<0}}$$

So; $\dfrac{\ln(1-R)}{\ln(1-p)} - 1 \le x < \dfrac{\ln(1-R)}{\ln(1-p)}$

$$\boxed{X = \left\lceil \frac{\ln(1-R)}{\ln(1-p)} - 1 \right\rceil}$$

When we want Geometric Random Variables that start from a, we have, $x = a, a+1, a+2, \ldots$

and $X = a + \left\lceil \dfrac{\ln(1-R)}{\ln(1-p)} - 1 \right\rceil$ (usually $a=1$ as asserted)

(Here, for $a \geq 1$, we have the mean $E(X) = \frac{1}{p} + a - 1$)

Example!! Generate 3 Geometric Random Variables with mean 4 on the Range $\{X \geq 1\}$

Answer!! $\frac{1}{p} = 4 \Rightarrow p = \frac{1}{4} = 0{,}25$ and $\frac{1}{\ln(1-p)} = \frac{1}{\ln 0{,}75} = -3{,}476$

$$X_i = 1 + \lceil -3{,}476 . \ln(1-R_i) - 1 \rceil$$

i	1	2	3
R_i	0,932	0,105	0,687
X_i	$10 = 1 + \lceil -3{,}476 . \ln(1-0{,}932) - 1 \rceil$	$1 = \ldots$	$5 = \ldots$

15. The weekly demand, X, for a slow-moving item has been found to be approximated well by a geometric distribution on the range $\{0, 1, 2, \dots\}$ with mean weekly demand of 2.5 items. Generate 7 values of X, demand per week, using random numbers from Table A.1. (*Hint*: For a geometric distribution on the range $\{q, q+1, \dots\}$ with parameter p, the mean is $1/p + q - 1$.)

15) $\frac{1}{p} = 2{,}5 \Rightarrow p = \frac{1}{2{,}5} = 0{,}4$; $\frac{1}{\ln(1-p)} = \frac{1}{\ln 0{,}6} = -1{,}958$

$$X_i = \lceil -1{,}958 \ln R_i - 1 \rceil$$

i	1	2	3	4	5	6	7
R_i	0,3157	0,0782	0,0906	0,1992	0,2992	0,2960	0,9661
X_i	2	4	4	3	2	2	0

$\lceil -1{,}958 \ln 0{,}3157 - 1 \rceil = \lceil 1{,}257 \rceil = 2$

(II) ACCEPTANCE-REJECTION TECHNIQUE

To get the idea of acceptance-rejection technique, we may start with the following simple example:

Simple Example:)

Let, we want to generate $X \sim \text{Uniform}(0{,}25;\ 1)$

We know that $X_i = 0{,}75 R_i + 0{,}25$ generates X_i, more efficiently. (We'll see why this is more efficient)

Alternatively, we may generate $R_i \sim \text{Uniform}(0, 1)$ and "Accept" as X_i if $R_i \geq 0{,}25$ (Namely in the Range)

Step 1 Generate R

Step 2 If $R \geq 0{,}25$, Accept $X = R$, GOTO step 3. o.w. GOTO Step 1.

Step 3 If another X is needed, GOTO step 1. o.w. STOP!

Note that;

$$P(a < R < b \mid 0{,}25 \le R \le 1) = \frac{P(a < R < b)}{P(0{,}25 \le R \le 1)} = \frac{b-a}{3/4}$$

But, $X \sim \text{Uniform}(1/4, 1)$

$$F(x) = \frac{x - 1/4}{1 - 1/4} = \frac{x - 1/4}{3/4} \text{ and } P(a < X < b) = F(b) - F(a) = \frac{b-a}{3/4}$$

So, this algorithm truely generates X_i!

Now, we are interested in the efficiency of this algorithm, which depends on "Number of Rejections"

Let Y: Random Numbers Required to generate First X_i

Remember, $Y \sim \text{Geometric}\left(p = \frac{3}{4}\right)$ → $p = P(0{,}25 \le R \le 1)$

$$E(Y) = \frac{1}{p} = \frac{1}{3/4} = \frac{4}{3}.$$

Let, we need 1000 random X_i. How many R_i, on the average, we need to generate 1000 X_i's?

Let: N: Random Numbers Required to generate 1000 X_i.

Remember, $N \sim \text{Negative Binomial}\left(r = 1000; p = \frac{3}{4}\right)$

Also Note that $\boxed{N = \sum_{i=1}^{1000} Y_i}$

$$E(N) = E\left[\sum_{i=1}^{1000} Y_i\right] = \sum_{i=1}^{1000} E(Y_i) = \sum_{i=1}^{1000} \frac{4}{3} = 1000 \cdot \frac{4}{3} = 1333$$

So, we "waste" 333 Random Numbers, which decreases the performance of the algorithm. The performance may be even much worse! (i.e. if $X_i \sim \text{Uniform}(0{,}9, 1)$)

Poisson Distribution

$$N \sim Poisson(\alpha)$$

$$p(n) = P(N=n) = \frac{e^{-\alpha} \cdot \alpha^n}{n!} \quad n = 0, 1, 2, \ldots$$

$$E(N) = \alpha \quad \text{and} \quad Var(N) = \alpha$$

Remember from Stochastic Processes, Number of events per unit time with rate α can be well modeled with Poisson Distribution.

Also remember the relationship with Poisson Process and Exponential Distribution. If $N(t)$ is a Poisson Process then T: "Interarrival Time" OR "Time until Next Event" follows an Exponential Distribution with the same rate:

$$\boxed{N(t) \sim Poisson(\alpha t) \iff T \sim Exponential(\alpha)}$$

Therefore, if A_j is the time of j^{th} arrival,

$$\boxed{N = n \iff A_1 + A_2 + \cdots + A_n \le 1 \le A_1 + A_2 + \cdots + A_n + A_{n+1}}$$

So, we have; $\sum_{i=1}^{n} A_i \le 1 < \sum_{i=1}^{n+1} A_i$ but $A_i = -\frac{1}{\alpha} \ln R_i$ (Generation of Exponential R.V.)

$$\sum_{i=1}^{n} -\frac{1}{\alpha} \ln R_i \le 1 < \sum_{i=1}^{n+1} -\frac{1}{\alpha} \ln R_i$$

$$\ln \prod_{i=1}^{n} R_i \ge -\alpha > \ln \prod_{i=1}^{n+1} R_i$$

$$\boxed{\prod_{i=1}^{n} R_i \ge e^{-\alpha} > \prod_{i=1}^{n+1} R_i}$$

This fact is the logic of the following algorithm:

Step 1: Set $n=0$; $P=1$

Step 2: Generate R_{n+1} and $P \leftarrow P \cdot R_{n+1}$

Step 3: If $P < e^{-\alpha}$ then accept $N=n$

o.w. Reject current n, $n \leftarrow n+1$ AND GOTO Step 2

Surely, this algorithm generates a single N. We loop it as many times as needed.

Note that, to generate a single $N=n$, we need $n+1$ random numbers R_i. So, on the average $E(N+1)=\alpha+1$ numbers are required to generate FIRST N and we must generate $r.(\alpha+1)$ Numbers to obtain r N's, which is quite inefficient if α is large.

Example: Let $X_i \sim Poisson(\alpha=0,2)$. Generate 3 X_i's.

First compute $\boxed{e^{-\alpha}=e^{-0,2}=0,8187.}$

1. $n=0, P=1$
2. $R_1=0,4357$, $P=1.R_1=0,4357$
3. $P=0,4357 < e^{-\alpha}=0,8187$, accept $N=0$

1. $n=0$, $P=1$
2. $R_1=0,4146$, $P=1 \cdot R_1$, $0,4146$
3. $P=0,4146 < e^{-\alpha}=0,8187$, accept $N=0$

i	1	2	3
N_i	0	0	2

1. $n=0, P=1$
2. $R_1=0,8353$, $P=1.0,8353=0,8353$
3. Reject $n=0$, $n=1$
2. $R_2=0,9952$ $P=R_1.R_2=0,8313$
3. Reject $n=1$, $n=2$
2. $R_3=0,8004$ $P=R_1.R_2.R_3=0,6654$
3. Accept $N=2$

16. In Exercise 15, suppose that the demand has been found to have a Poisson distribution with mean 2.5 items per week. Generate 4 values of X, demand per week, using random numbers from Table A.1. Discuss the differences between the geometric and the Poisson distributions.

16) $e^{-\alpha} = e^{-2,5} = 0{,}082 \quad N \sim \text{Poisson}(\alpha = 2{,}5)$

n	Step 1 0	0	$\boxed{1}$	Step 1 0	0	1	$\boxed{2}$
R_{n+1}	–	0,1093	0,4756	–	0,7787	0,1648	0,4397
$P_{n+1} = P_n \cdot R_{n+1}$	1	0,1093	0,0520	1	0,7787	0,1283	0,0564
A/R	–	R	A	–	R	R	A

n	Step 1 0	0	1	2	3	$\boxed{4}$	Step 1 0	$\boxed{0}$
R_{n+1}	–	0,9667	0,6791	0,7926	0,2615	0,4651	–	0,0258
$P_{n+1} = P_n \cdot R_{n+1}$	1	0,9667	0,6429	0,5096	0,1333	0,0612	1	0,0258
A/R	–	R	R	R	R	A	–	A

The Generated Random Numbers are 1 ; 2 ; 4 and 0.

Note that, we generated 11 R_i's to Generate 4 N_i's.

Average Number of R_i's to generate 4 N_i's is;

$r \cdot E(N+1) = r \cdot (\alpha + 1) = 4 \cdot (2{,}5 + 1) = \underline{\underline{14}}$

Normal Approximation;

$\boxed{Z = \frac{N - \alpha}{\sqrt{\alpha}} \Rightarrow N_i = \lceil \alpha + \sqrt{\alpha}\, Z_i - 0{,}5 \rceil}$ can be used to generate Poisson Random Variables for α is large (practically $\alpha \geq 15$)

Remember, we have seen how to generate Z_i.

Nonstationary Poisson Process (NSPP)

Remember, a NSPP is an arrival process with an arrival rate that varies with time. The following algorithm, which is also called "thinning" algorithm can be used to generate interarrival times from a NSPP with arrival rate $\lambda(t)$, $0 \le t \le T$

Step 1. Let $\lambda^* = \max_{0 \le t \le T} \lambda(t)$ and set $t=0$, $i=1$

Step 2. Generate $E \sim \text{Exponential}(\lambda^*)$ and $t \leftarrow t+E$

Step 3. Generate $R \sim \text{Uniform}(0;1)$

If $R \le \dfrac{\lambda(t)}{\lambda^*}$ then $T_i = t$ and $i \leftarrow i+1$

Step 4. GOTO Step 2

This algorithm works for any integrable arrival rate function. However, we'll only see the case that Arrival rate function is a piece-wise-constant function.

Example

t (min)	Mean time between Arrivals (min)	Arrival Rate: $\lambda(t)$ (arrivals/min)
0	15	1/15
60	12	1/12
120	7	1/7
180	5	$\boxed{1/5 = \lambda^*}$
240	8	1/8
300	10	1/10
360	15	1/15
420	20	1/20
480	20	1/20

Generate first two arrivals.

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Step 1: $\lambda^* = \max\limits_{0 \le t \le T} \lambda(t) = \frac{1}{5}$; $t=0$; $i=1$

Step 2: $R = 0{,}2130$; $E = -5 \ln(0{,}2130) = 13{,}13$; $t = 0 + 13{,}13 = 13{,}13$

Step 3: $R = 0{,}8830 \not\le \frac{\lambda(13{,}13)}{\lambda^*} = \frac{1/15}{1/5} = \frac{1}{3}$ do NOT generate on interval.

Step 4: GOTO Step 2

Step 2: $R = 0{,}5530$; $E = -5 \ln(0{,}5530) = 2{,}96$; $t = 13{,}13 + 2{,}96 = 16{,}09$

Step 3: $R = 0{,}0240 \le \frac{\lambda(16{,}09)}{\lambda^*} = \frac{1/15}{1/5} = \frac{1}{3}$

$\boxed{T_1 = t = 16{,}09}$ $\quad i = i+1 = 2$

Step 4: GOTO Step 2

Step 2: $R = 0{,}0001$; $E = -5 \ln(0{,}0001) = 46{,}05$; $t = 16{,}09 + 46{,}05 = 62{,}14$

Step 3: $R = 0{,}1443 \le \frac{\lambda(62{,}14)}{\lambda^*} = \frac{1/12}{1/5} = \frac{5}{12} = 0{,}4167$

$\boxed{T_2 = t = 62{,}04}$ $\quad i = i+1 = 3$

29. Let time $t = 0$ correspond to 6 A.M., and suppose that the arrival rate (in arrivals per hour) of customers to a breakfast restaurant that is open 6–9 A.M. is

$$\lambda(t) = \begin{cases} 30, & 0 \le t < 1 \\ 45, & 1 \le t < 2 \\ 20, & 2 \le t \le 4 \end{cases}$$

Derive a thinning algorithm for generating arrivals from this NSPP.

29) Step 1: $\lambda^* = 45$ because given numbers are directly rate.

$t=0$ $\quad i=1$

$E_i = -\frac{1}{45} \ln R_i$

0,5766 0,2379 0,1747 ...

Let R be given: 0,7788 0,8894 0,6394 0,4755 0,0260

For ease of calculation, we may take unit time as minutes. Then,

$$\lambda(t) = \begin{cases} \frac{30}{60} = 0{,}5 \text{ arr/min} & 0 \le t < 60 \\ \frac{45}{60} = 0{,}75 \text{ arr/min} & 60 \le t < 120 \\ \frac{20}{60} = 0{,}33 \text{ arr/min} & 120 \le t \le 240 \end{cases}$$

$\lambda^* = 0{,}75$ $\quad t = 0 \quad i = 1 \quad E_i = -\frac{4}{3} \ln R_i$

Step 2: $E = -\frac{4}{3} \ln 0{,}7788 = 0{,}34 \qquad t = 0{,}34 + 0 = 0{,}34$

Step 3 $R = 0{,}8894 \not\le \frac{\lambda(0{,}34)}{\lambda^*} = \frac{0{,}50}{0{,}75} = 0{,}6667$

Do NOT generate an interval

Step 4 GOTO step 2

Step 2 $E = -\frac{4}{3} \ln 0{,}6394 = 0{,}59 \qquad t = 0{,}34 + 0{,}59 = 0{,}93$

Step 3 $R = 0{,}6755 \le \frac{\lambda(0{,}93)}{\lambda^*} = 0{,}6667$; $\boxed{T_1 = t = 0{,}93}$ $i = 2$

Step 4 GOTO Step 2

Step 2 $E = -\frac{4}{3} \ln 0{,}0260 = 4{,}85$; $t = 0{,}93 + 4{,}85 = 5{,}78$

Step 3 $R = 0{,}1767 \le \frac{\lambda(5{,}78)}{\lambda^*} = 0{,}6667$; $\boxed{T_2 = 5{,}78}$

Note that, when the period is that of λ^*, all arrivals will be Accepted. The idea is basically to lower the Arrivals proportional to λ^*. Namely, an arrival will be Accepted with probability $\frac{\lambda(t)}{\lambda^*}$.

(III) CONVOLUTION METHOD

The probability distribution of a sum of two or more independent random variables is called a convolution of the distribution of the original variables.

The convolution method thus refers to adding together two or more random variables to obtain a new random variable with the desired distribution.

Erlang Distribution

$$X \sim \text{Erlang}(k, \theta)$$

$$E(X) = \frac{1}{k\theta} \qquad Var(X) = \frac{1}{k\theta^2}$$

Remember, Erlang distribution is sum of k i.i.d Exponential Random Variables with mean $\frac{1}{k\theta}$.

let $y_i \sim \text{Exponential}(k\theta)$

Then; $X = \sum_{i=1}^{k} y_i$ but $y_i = -\frac{1}{k\theta} \ln R_i$

$$\boxed{X = \sum_{i=1}^{k} -\frac{1}{k\theta} \ln R_i = -\frac{1}{k\theta} \ln \left(\prod_{i=1}^{k} R_i \right)}$$

Negative Binomial Distribution

$$X \sim \text{Negative Binomial}(r; p)$$

$$E(X) = \frac{r}{p} \qquad Var(X) = \frac{r(1-p)}{p^2}$$

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Remember, $Y \sim \text{Geometric}(p) \Rightarrow Y$ is the number of trials to obtain FIRST success and X is Number of trials to obtain r^{th} success.

Then, $\boxed{X = \sum_{i=1}^{r} Y_i}$ $Y_i \sim \text{Geometric}(p)$

$X \sim \text{Negative Binomial}(r;p)$

14. Develop a method for generating values from a negative binomial distribution with parameters p and k, as described in Section 5.3. Generate 3 values when $p = 0.8$ and $k = 2$. [*Hint*: Think about the definition of the negative binomial as the number of Bernoulli trials until the kth success.]

14) We have learned how to generate Geometric R.V's by Inverse transformation technique. Alternatively, we may generate them by Acceptance - Rejection Technique.

Step 0 : set $X = 0$, set r, set $i = 1$, set p

Step 1 : Set $n = 1$

Step 2 : Generate R

Step 3 : If $R \le p$ Accept $Y = n$

o.w, $n \leftarrow n+1$, GOTO Step 2

Step 4 : $X \leftarrow X + Y$

Step 5 : If $i = r$, stop

o.w, $i \leftarrow i+1$ and GOTO step 1

First value

$X = 0;\ r = 2;\ i = 1;\ p = 0.8$

$n = 1$

$R = 0.8253 \nleq 0.8$

$n = 2$

$R = 0.0733 \le 0.8;\ Y = 2$

$X = 0 + 2 = 2$

$i = 1 \ne 2$, GOTO Step 1, $i = 2$

$n = 1$

$R = 0.7878 \le 0.8;\ Y = 1$

$\boxed{X = 2 + 1 = 3}$

$i = 2 = 2$ STOP