

MANAGEMENT SCIENCE
 LECTURE NOTES

 CHAPTERS
 283

Linear Programming (LP)

In linear programming questions, we either

- (i) Maximize Profit OR
- (ii) Minimize Cost.

we follow these steps to formulate an LP problem

(i) Identify Decision Variables

Decision Variables are easily identified from the aim of the question (usually at last sentence). Note that, each decision variable has a unit profit OR a unit cost.

(ii) Formulate the Objective Function

Let the unit profit (OR cost) of the decision variables is c_1, c_2, \dots, c_k . The objective function is,

$$\text{maximize} \quad Z = c_1 x_1 + c_2 x_2 + \dots + c_k x_k \\ (\text{OR minimize})$$

(iii) Establish model constraints

In general, maximization problems have AVAILABLE resources which limits profit and minimization problems have minimum REQUIREMENTS which bounds cost.

Constraints have form $a_1 x_1 + a_2 x_2 + \dots + a_k x_k \geq b_j$ or

$$a_1 x_1 + a_2 x_2 + \dots + a_k x_k \leq b_j \text{ or}$$

$$a_1 x_1 + a_2 x_2 + \dots + a_k x_k = b_j$$

4. The Pinewood Furniture Company produces chairs and tables from two resources—labor and wood. The company has 80 hours of labor and 36 pounds of wood available each day. Demand for chairs is limited to 6 per day. Each chair requires 8 hours of labor and 2 pounds of wood, whereas a table requires 10 hours of labor and 6 pounds of wood. The profit derived from each chair is \$400 and from each table, \$100. The company wants to determine the number of chairs and tables to produce each day in order to maximize profit.
 - a. Formulate a linear programming model for this problem.
 - b. Solve this model by using graphical analysis.
5. In Problem 4, how much labor and wood will be unused if the optimal numbers of chairs and tables are produced?
6. In Problem 4, explain the effect on the optimal solution of changing the profit on a table from \$100 to \$500.

4)

a)

	Labor	Wood	PROFIT
(x_1) Chairs	8	2	400
(x_2) Tables	10	6	100
AVAILABLE	80	36	

Maximize $Z = 400x_1 + 100x_2$

subject to $8x_1 + 10x_2 \leq 80$
 $2x_1 + 6x_2 \leq 36$

$$x_1 \leq 6$$

$$x_1, x_2 \geq 0$$

b) (i) $8x_1 + 10x_2 = 80$ (ii) $2x_1 + 6x_2 = 36$ (iii) $x_1 = 6$
 $x_1 = 0 \Rightarrow x_2 = 8$ $x_1 = 0 \Rightarrow x_2 = 6$
 $x_2 = 0 \Rightarrow x_1 = 10$ $x_2 = 0 \Rightarrow x_1 = 18$

(i) & (ii)

$$\begin{aligned} 8x_1 + 10x_2 &= 80 \\ -4(2x_1 + 6x_2) &= -4(36) \\ \hline 8x_1 + 10x_2 &= 80 \\ -8x_1 - 24x_2 &= -144 \\ \hline -14x_2 &= -64 \end{aligned}$$

$$x_2 = 4,57$$

$$2x_1 + 6 \cdot 4,57 = 36$$

$$2x_1 = 8,57$$

$$x_1 = 4,28$$

(i) & (iii)

$$x_1 = 6$$

$$8x_1 + 10x_2 = 80$$

$$8 \cdot 6 + 10x_2 = 80$$

$$10x_2 = 32$$

$$x_2 = 3,2$$

(ii) & (iii)

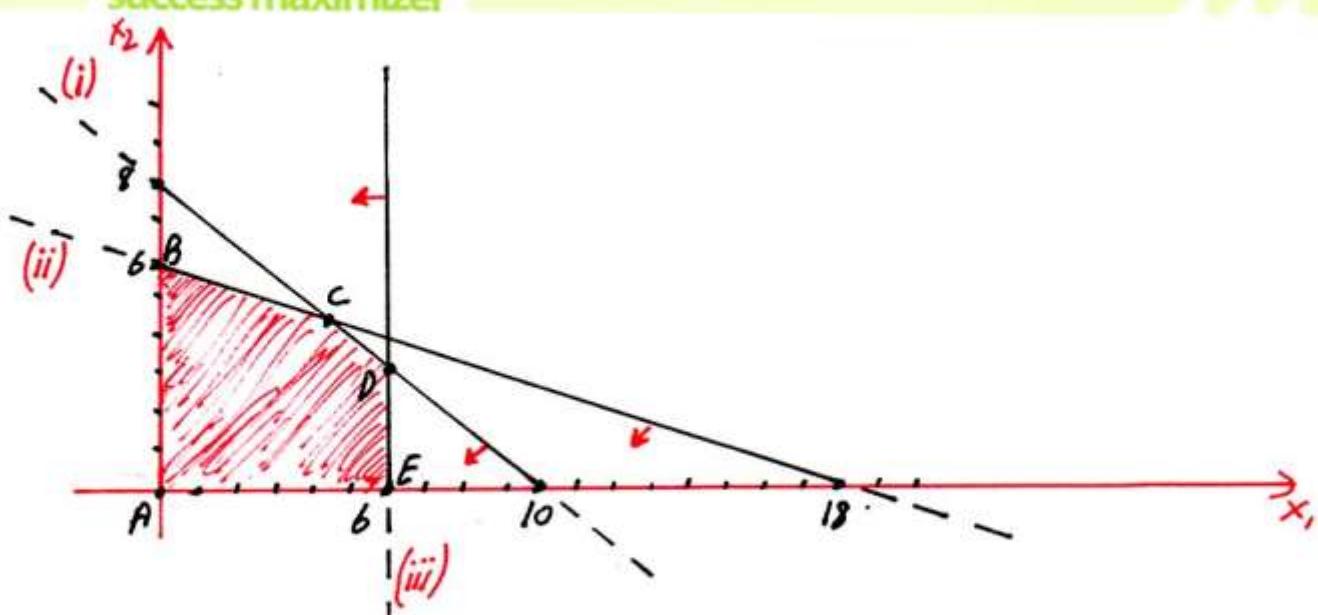
$$x_1 = 6$$

$$2x_1 + 6x_2 = 36$$

$$2 \cdot 6 + 6x_2 = 36$$

$$6x_2 = 24$$

$$x_2 = 4$$



x_1	x_2	$Z = 400x_1 + 100x_2$
A (0 ; 0)		$z = 400 \cdot 0 + 100 \cdot 0 = 0$
B (0 ; 6)		$z = 400 \cdot 0 + 100 \cdot 6 = 600$
C (4,28 ; 4,57)		$z = 400 \cdot 4,28 + 100 \cdot 4,57 = 2169$
D (6 ; 3,2)		$z = 400 \cdot 6 + 100 \cdot 3,2 = \cancel{2720} \rightarrow \text{Maximum}$
E (6 ; 0)		$z = 400 \cdot 6 + 100 \cdot 0 = 2400$

Maximum profit is 2720 when 6 chairs and 3,2 tables are produced.

5) Labor Used: $8 \cdot 6 + 10 \cdot 3,2 = 80$

Unused Labor = $80 - 80 = 0$

Wood Used: $2 \cdot 6 + 6 \cdot 3,2 = 31,2$

Unused Wood = $36 - 31,2 = 4,8$

6) $x_1 \quad x_2 \quad Z = 400x_1 + 500x_2$

A (0 ; 0) $z = 400 \cdot 0 + 500 \cdot 0 = 0$

B (0 ; 6) $z = 400 \cdot 0 + 500 \cdot 6 = 3000$

C (4,28 ; 4,57) $z = 400 \cdot 4,28 + 500 \cdot 4,57 = 3997$

D (6 ; 3,2) $z = 400 \cdot \cancel{6} + 500 \cdot 3,2 = \cancel{4000} \rightarrow \text{Maximum}$

E (6 ; 0) $z = 400 \cdot 6 + 500 \cdot 0 = 2400$

Maximum profit increases to 4000 but x_1, x_2 values do NOT change. ③

2. The Munchies Cereal Company makes a cereal from several ingredients. Two of the ingredients, oats and rice, provide vitamins A and B. The company wants to know how many ounces of oats and rice it should include in each box of cereal to meet the minimum requirements of 48 milligrams of vitamin A and 12 milligrams of vitamin B while minimizing cost. An ounce of oats contributes 8 milligrams of vitamin A and 1 milligram of vitamin B, whereas an ounce of rice contributes 6 milligrams of A and 2 milligrams of B. An ounce of oats costs \$0.05, and an ounce of rice costs \$0.03.

 - Formulate a linear programming model for this problem.
 - Solve this model by using graphical analysis.

3. What would be the effect on the optimal solution in Problem 2 if the cost of rice increased from \$0.03 per ounce to \$0.06 per ounce?

	Vitamin A	Vitamin B	COST
(x ₁) Oats	8	1	0,05
(x ₂) Rice	6	2	0,03
REQUIREMENT	48	12	

$$\text{Minimize} \quad z = 0.05x_1 + 0.03x_2$$

$$\text{subject to } 8x_1 + 6x_2 \geq 48$$

$$x_1 + 2x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

$$b) (i) 8x_1 + 6x_2 = 48$$

$$x_1 = 0 \Rightarrow x_2 = 8$$

$$x_2 \geq 0 \Rightarrow x_1 = 6$$

$$(ii) x_1 + 2x_2 = 12$$

$$x_1 = 0 \Rightarrow x_2 = 6$$

$$x_3 = 0 \Rightarrow x_1 = 12$$

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$$8x_1 + 6x_2 = 48$$

$$-3/ x_1 + 2x_2 = 12$$

$$\underline{8x_1 + 6x_2 = 48}$$

$$-3x_1 - 6x_2 = -36$$

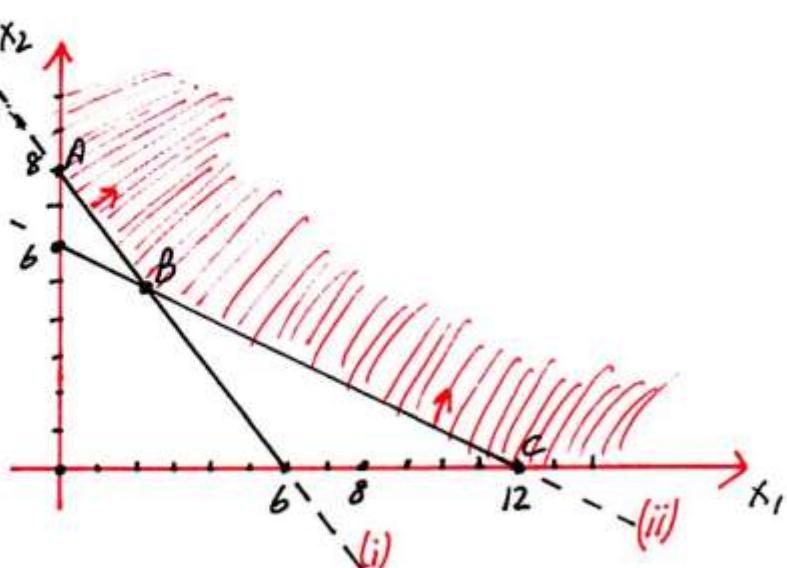
$$5x_1 = 12$$

$$\boxed{x_1 = 2, 4}$$

$$2 \cdot 4 + 2 \times 2 = 12$$

$$2 \times 2 = 9.6$$

$$x_2 = 4,8$$



x_1	x_2	$Z = 0.05x_1 + 0.03x_2$
A (0 ; 8)		$Z = 0.05 \cdot 0 + 0.03 \cdot 8 = 0.24$ <i>Minimum</i>
B (2.4 ; 4.8)		$Z = 0.05 \cdot 2.4 + 0.03 \cdot 4.8 = 0.264$
C (12 ; 0)		$Z = 0.05 \cdot 12 + 0.03 \cdot 0 = 0.6$

Minimum Cost is 0.24 when NO oats and 8ounce Rice are used.

3)	x_1	x_2	$Z = 0.05x_1 + 0.06x_2$
A (0 ; 8)			$Z = 0.05 \cdot 0 + 0.06 \cdot 8 = 0.48$
B (2.4 ; 4.8)			$Z = 0.05 \cdot 2.4 + 0.06 \cdot 4.8 = 0.408$ <i>Minimum</i>
C (12 ; 0)			$Z = 0.05 \cdot 12 + 0.06 \cdot 0 = 0.6$

Optimum solution completely changes. Minimum Cost is 0.408 when 2.4 ounce Oats and 4.8 ounce Rice are used.

11. Solve the following linear programming model graphically:

$$\text{maximize } Z = 1.5x_1 + x_2$$

subject to

$$(i) \quad x_1 \leq 4$$

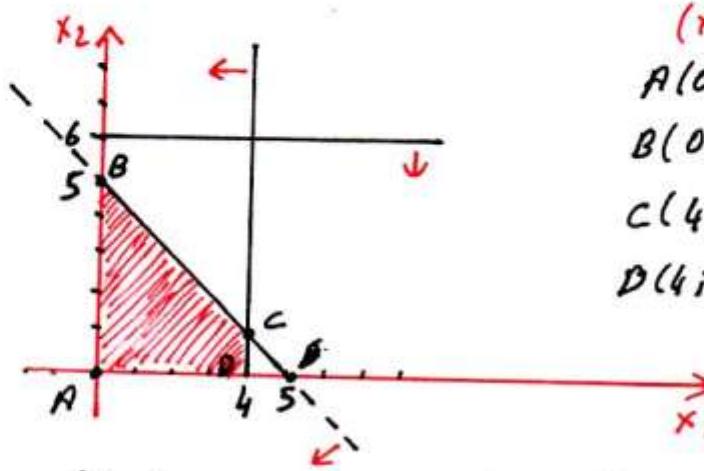
$$(ii) \quad x_2 \leq 6$$

$$(iii) \quad x_1 + x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

12. Transform the model in Problem 11 into standard form and indicate the value of the slack variables at each corner point solution.

11)



(x_1, x_2)	$Z = 1.5x_1 + x_2$
A(0; 0)	$Z = 1.5 \cdot 0 + 0 = 0$
B(0; 5)	$Z = 1.5 \cdot 0 + 5 = 5$
C(4; 1)	$Z = 1.5 \cdot 4 + 1 = 7$
D(4; 0)	$Z = 1.5 \cdot 4 + 0 = 6$ <i>maximum</i>

Maximum profit is 7 when $x_1 = 4$ and $x_2 = 1$

12) **Slack Variables:** Add s_i : slack variable to a \leq constraint to obtain a linear equation " $=$ ". Slack variables show unused AVAILABLE sources for the corresponding constraints. We have;

standard form:

$$(i) x_1 + s_1 = 4$$

$$(ii) x_2 + s_2 = 6$$

$$(iii) x_1 + x_2 + s_3 = 5$$

$$A(0; 0)$$

$$B(0; 5)$$

$$C(4; 1)$$

$$D(4; 0)$$

$$s_1 = 4$$

$$s_2 = 1$$

$$s_3 = 0$$

$$s_1 = 0$$

$$s_2 = 5$$

$$s_3 = 1$$

$$s_1 = 0$$

$$s_2 = 6$$

$$s_3 = 0$$

39. Solve the following linear programming model graphically and explain the solution result:

$$\text{minimize } Z = \$3,000x_1 + 1,000x_2$$

subject to

$$60x_1 + 20x_2 \geq 1,200$$

$$10x_1 + 10x_2 \geq 400$$

$$40x_1 + 160x_2 \geq 2,400$$

$$x_1, x_2 \geq 0$$

$$(i) 60x_1 + 20x_2 = 1200 \quad (ii) 10x_1 + 10x_2 = 400 \quad (iii) 40x_1 + 160x_2 = 2400$$

$$3x_1 + x_2 = 60$$

$$x_1 + x_2 = 40$$

$$x_1 + 4x_2 = 60$$

$$x_1 = 0 \Rightarrow x_2 = 60$$

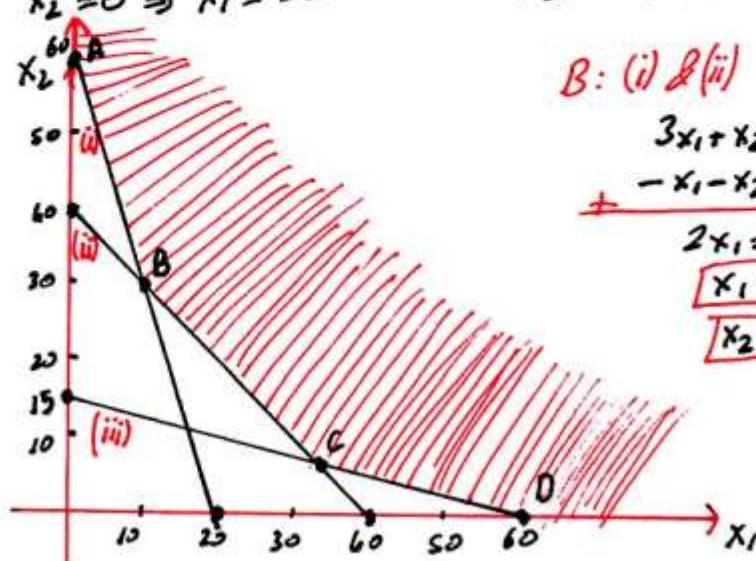
$$x_1 = 0 \Rightarrow x_2 = 40$$

$$x_1 = 0 \Rightarrow x_2 = 15$$

$$x_2 = 0 \Rightarrow x_1 = 20$$

$$x_2 = 0 \Rightarrow x_1 = 40$$

$$x_2 = 0 \Rightarrow x_1 = 60$$



B: (i) & (ii)

$$\begin{array}{r} 3x_1 + x_2 = 60 \\ -x_1 - x_2 = 40 \\ \hline 2x_1 = 20 \end{array}$$

$$\begin{array}{l} x_1 = 10 \\ x_2 = 30 \end{array}$$

$$\begin{array}{r} -x_1 - x_2 = -40 \\ x_1 + 4x_2 = 60 \\ \hline 3x_2 = 20 \end{array}$$

$$\begin{array}{l} x_2 = 6.67 \\ x_1 + 6.67 = 60 \end{array}$$

$$\boxed{x_1 = 33,33}$$

$$x_1 \quad x_2 \quad z = 3000x_1 + 1000x_2$$

$$A(0; 60) \quad z = 3000 \cdot 0 + 1000 \cdot 60 = \underline{\underline{60000}}$$

$$B(10; 30) \quad z = 3000 \cdot 10 + 1000 \cdot 30 = \underline{\underline{60000}} \rightarrow \text{Minimum}$$

$$C(33,33; 6,67) \quad z = 3000 \cdot 33,33 + 1000 \cdot 6,67 = 166666$$

$$D(60; 0) \quad z = 3000 \cdot 60 + 1000 \cdot 0 = 180000$$

when two minimum values are equal, the line segment has all the same solutions. The points in $|AB|$ all have minimum value of 60000.

b) Transform the model into standard form and indicate the value of excess (surplus) variables at each corner point solution.

Excess variables: Subtract e_i : excess (surplus) variable to a \geq constraint to obtain a linear equation " $=$ ". Excess variables show extra REQUIREMENT added to the corresponding constraint.

We have;

	x_1	x_2	x_1	x_2	x_1	x_2
standard form	$A(0; 60)$	$B(10; 30)$	$C(33,33; 6,67)$	$D(60; 0)$	x_1	x_2

$$(i) \quad 60x_1 + 20x_2 - e_1 = 1200 \quad e_1 = 0 \quad e_1 = 0 \quad e_1 = 933,32 \quad e_1 = 2600$$

$$(ii) \quad 10x_1 + 10x_2 - e_2 = 400 \quad e_2 = 200 \quad e_2 = 0 \quad e_2 = 0 \quad e_2 = 200$$

$$(iii) \quad 40x_1 + 160x_2 - e_3 = 2400 \quad e_3 = 7200 \quad e_3 = 2800 \quad e_3 = 0 \quad e_3 = 0$$

For example at Point B, (i) and (ii) are just satisfied and 2800 units of extra constraint (iii) is obtained.

31. Angela and Bob Ray keep a large garden in which they grow cabbage, tomatoes, and onions to make two kinds of relish—chow-chow and tomato. The chow-chow is made primarily of cabbage, whereas the tomato relish has more tomatoes than chow-chow. Both relishes include onions, and negligible amounts of bell peppers and spices. A jar of chow-chow contains 8 ounces of cabbage, 3 ounces of tomatoes, and 3 ounces of onions, whereas a jar of tomato relish contains 6 ounces of tomatoes, 6 ounces of cabbage, and 2 ounces of onions. The Rays grow 120 pounds of cabbage, 90 pounds of tomatoes, and 45 pounds of onions each summer. The Rays can produce no more than 24 dozen jars of relish. They make \$2.25 in profit from a jar of chow-chow and \$1.95 in profit from a jar of tomato relish. The Rays want to know how many jars of each kind of relish to produce to generate the most profit.

a. Formulate a linear programming model for this problem.

b. Solve this model graphically.

	Cabbage	Tomato	Onions	PROFIT
(x_1) Chow-Chow	8	3	3	2.25
(x_2) Tomato	6	6	2	1.95
AVAILABLE	120	90	45	

Maximize $Z = 2.25x_1 + 1.95x_2$

subject to

$$8x_1 + 6x_2 \leq 120$$

$$3x_1 + 6x_2 \leq 90$$

$$3x_1 + 2x_2 \leq 45$$

$$x_1 + x_2 \leq 24$$

$$x_1, x_2 \geq 0$$

$$(iv) x_1 + x_2 = 24$$

$$x_1 = 0 \Rightarrow x_2 = 24$$

$$x_2 = 0 \Rightarrow x_1 = 24$$

b) (i) $8x_1 + 6x_2 = 120$ (ii) $3x_1 + 6x_2 = 90$ (iii) $3x_1 + 2x_2 = 45$

$x_1 = 0 \Rightarrow x_2 = 20$ $x_1 = 0 \Rightarrow x_2 = 15$ $x_1 = 0 \Rightarrow x_2 = 22.5$

$x_2 = 0 \Rightarrow x_1 = 15$ $x_2 = 0 \Rightarrow x_1 = 30$ $x_2 = 0 \Rightarrow x_1 = 15$

c: (i) & (ii)

$$8x_1 + 6x_2 = 120$$

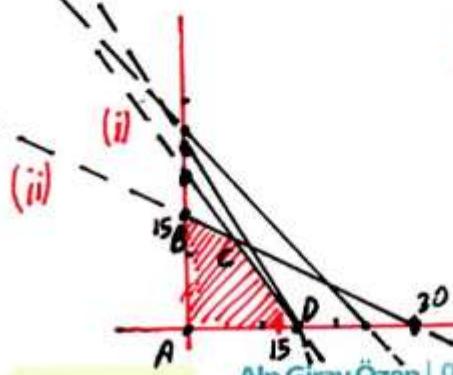
$$3x_1 + 6x_2 = 90$$

$$5x_1 = 30$$

$$x_1 = 6$$

$$3.6 + 6x_2 = 90$$

$$x_2 = 12$$



$$x_1 \quad x_2 \quad z = 2,25x_1 + 1,95x_2$$

$$A(0; 0) \quad z = 2,25 \cdot 0 + 1,95 \cdot 0 = 0$$

$$B(0; 15) \quad z = 2,25 \cdot 0 + 1,95 \cdot 15 = 29,25$$

$$C(6; 12) \quad z = 2,25 \cdot 6 + 1,95 \cdot 12 = \underline{36,3} \quad \text{maximum}$$

$$D(15; 0) \quad z = 2,25 \cdot 15 + 1,95 \cdot 0 = 33,75$$

Maximum profit is 36,3 when 6 units chow-chow and 12 units tomato jar is produced.

c) Analyze the optimum point in terms of unused resources.

$$S_1 = -8,6 - 6 \cdot 12 + 120 = 0 \quad \text{All cabbage is used}$$

$$S_2 = -3,6 - 6 \cdot 12 + 90 = 0 \quad \text{All Tomato is used}$$

$$S_3 = -3,6 - 2 \cdot 12 + 45 = 3 \quad 3 \text{ ounces of onion still available}$$

$$S_4 = -6 - 12 + 24 = 6 \quad 6 \text{ dozen of jars can be } \frac{\text{still}}{\text{produced}}$$

22. Gillian's Restaurant has an ice-cream counter where it sells two main products, ice cream and frozen yogurt, each in a variety of flavors. The restaurant makes one order for ice cream and yogurt each week, and the store has enough freezer space for 115 gallons total of both products. A gallon of frozen yogurt costs \$0.75 and a gallon of ice cream costs \$0.93, and the restaurant budgets \$90 each week for these products. The manager estimates that each week the restaurant sells at least twice as much ice cream as frozen yogurt. Profit per gallon of ice cream is \$4.15, and profit per gallon of yogurt is \$3.60.

Formulate a linear programming model for this problem.

22) a) Maximize $\bar{z} = 4,15x_1 + 3,60x_2$

subject to $0,75x_1 + 0,93x_2 \leq 90$

$$x_1 + x_2 \leq 115$$

$$x_1 \geq 2x_2 \Rightarrow x_1 - 2x_2 \geq 0$$

$$x_1 \geq 0, x_2 \geq 0$$

Sensitivity Analysis

Remember questions 6 & 3. We were asked the effect of the change in unit profit(cost) to the optimal solution. At question 6, the change was only at the optimum value. At question 3, the solution was completely changed.

This is a part of the search of sensitivity analysis. Every other things remain constant, what is the allowable change of a unit profit(or cost) that keeps the solution set the same?

Secondly, we will analyze the change in AVAILABLE sources (or minimum REQUIREMENTS). Here, the solution always changes as well as the optimum value. However, within some limits of right hand side of a constraint, the solution mix (nonzero elements of the solution space) do NOT change.

In this second application, another interest is the "Shadow Price" (Dual Value) of the constraint. For a max. problem, shadow price is defined as the Marginal value of one additional unit of resource. Keep in mind that shadow prices are valid only for the range of rhs of the constraint. Then, without resolving the whole problem, we will be able to calculate new profit within change in range.

* Consider the following max. problem:

	Labor	Clay	PROFIT
(x ₁) Bowl	1	4	40
(x ₂) Mug	2	3	50
AVAILABLE	40	120	

The LP formulation is;

$$\text{maximize } Z = 40x_1 + 50x_2$$

$$\text{subject to } \begin{aligned} x_1 + 2x_2 &\leq 40 \\ 4x_1 + 3x_2 &\leq 120 \end{aligned}$$

The graphical solution of the problem is;

$$(i) x_1 + 2x_2 = 40$$

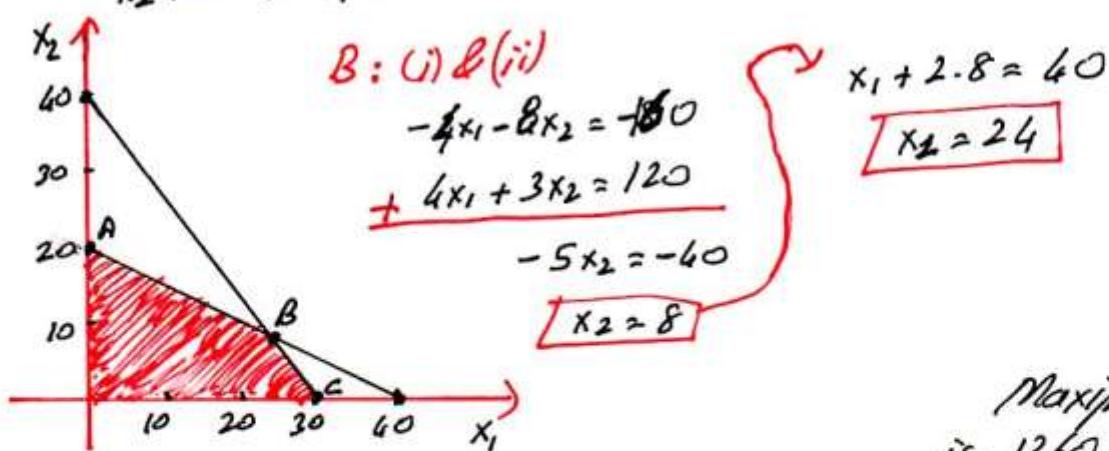
$$x_1 = 0 \Rightarrow x_2 = 20$$

$$x_2 = 0 \Rightarrow x_1 = 40$$

$$(ii) 4x_1 + 3x_2 = 120$$

$$x_1 = 0 \Rightarrow x_2 = 40$$

$$x_2 = 0 \Rightarrow x_1 = 30$$



$$\begin{array}{ll} x_1 & x_2 \\ \hline A(0; 20) & z = 40 \cdot 0 + 50 \cdot 20 = 1000 \\ B(24; 8) & z = 40 \cdot 24 + 50 \cdot 8 = 1360 \\ C(30; 0) & z = 40 \cdot 30 + 50 \cdot 0 = 1200 \end{array}$$

Maximum profit
is 1360 when
24 units of bowl
and 8 units of
mugs are produced.

maximum

We will learn how to interpret the computer output of solution and how to make sensitivity analysis:

Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X1	24	0	40	25	66,6667
X2	8	0	50	30	80
Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Labor	16	0	40	30	80
Clay	6	0	120	60	160

The upper part of the table shows the analysis' first unit profits. The "Value columns" shows the solution;

$$x_1 = 24 \quad x_2 = 8$$

And the slack variables are;

$$s_1 = 0 \quad s_2 = 0 \quad \text{from the second part.}$$

The "Original Value" column shows the unit profits;

$$Z = 40x_1 + 50x_2$$

Let the objective function is;

$$Z = c_1 x_1 + c_2 x_2$$

The "Lower Bound" and "Upper Bound" values mean that;

(i) For $25 \leq c_1 \leq 66,67$, optimum solution $x_1 = 24$ and $x_2 = 8$ remains the same (keeping $c_2 = 50$ constant) but the optimum value changes.

(ii) For $30 \leq c_2 \leq 80$, optimum solution $x_1 = 24$ and $x_2 = 8$ remains the same (keeping $c_1 = 40$ constant) but the optimum value changes.

For example, keeping mg' s ^{unit} profit at 50 constant, let Bowl's unit profit is increased from 40 to 55. Then, our new LP is;

Maximize $Z = 55x_1 + 50x_2$
 subject to $x_1 + 2x_2 \leq 40$
 $4x_1 + 3x_2 \leq 120$

Since 55 is in the interval $(25; 66, 67)$, the solution of this new LP is;

maximum profit is $55 \cdot 24 + 50 \cdot 8 = 1720$
 when $x_1 = 24$ and $x_2 = 8$.

Likewise; let, keeping mg' s ^{Bowl's} unit profit at 40 constant, let mg' s profit is decreased from 50 to 35. Then, our new LP is;

Maximize $Z = 40x_1 + 35x_2$
 subject to $x_1 + 2x_2 \leq 40$
 $4x_1 + 3x_2 \leq 120$

Since 35 is in the interval $(130; 80)$, the solution of this new LP is;

Maximum profit is $40 \cdot 24 + 35 \cdot 8 = 1240$
 when $x_1 = 24$ and $x_2 = 8$.

Now, consider the lower part of the table which shows analysis for constraints,

$$(\text{Labor}) \quad x_1 + 2x_2 \leq 40$$

$$(\text{Clay}) \quad 4x_1 + 3x_2 \leq 120$$

The "Original Value" column shows the right hand side (rhs) let the labor hours is increased from 40 to 60. New LP and its graphical solution is as follows;

$$\text{Maximize } z = 60x_1 + 50x_2$$

$$\text{subject to } x_1 + 2x_2 \leq 60$$

$$4x_1 + 3x_2 \leq 120$$

$$(i) \quad x_1 + 2x_2 = 60$$

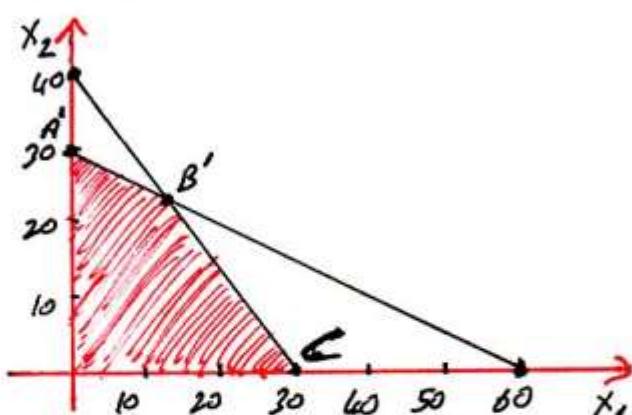
$$x_1 = 0 \Rightarrow x_2 = 30$$

$$x_2 = 0 \Rightarrow x_1 = 60$$

$$(ii) \quad 4x_1 + 3x_2 = 120$$

$$x_1 = 0 \Rightarrow x_2 = 40$$

$$x_2 = 0 \Rightarrow x_1 = 30$$



$$A' (0; 30) \quad z = 60 \cdot 0 + 50 \cdot 30 = 1500$$

$$B' (12; 24) \quad z = 60 \cdot 12 + 50 \cdot 24 = 1680$$

$$C' (30; 0) \quad z = 60 \cdot 30 + 50 \cdot 0 = 1200$$

maximum

$$B': (i) \& (ii)$$

$$-4x_1 - 8x_2 = -240$$

~~$$+ 4x_1 + 3x_2 = 120$$~~

$$-5x_2 = -120$$

$$x_2 = 24$$

$$x_1 + 2 \cdot 24 = 60$$

$$x_1 = 12$$

Maximum profit is 1680 when 12 Bowls and 24 Mugs are produced.

In the solution of an LP, (in general) the variables that take 0 are called Nonbasic Variables whereas the variables that are nonzero are called Basic Variables.

Comparison of two LP's give us;

Available Labor is 60
(Old LP)

$$Z = 1360 \text{ when}$$

$$x_1 = 24; x_2 = 8;$$

$$s_1 = 0; s_2 = 0$$

Available Labor is 60
(New LP)

$$Z = 1680 \text{ when}$$

$$x_1 = 12; x_2 = 24;$$

$$s_1 = 0; s_2 = 0 \text{ (check s; values.)}$$

As we follow, the solution has completely changed. However, the solution mix (basic & Nonbasic Variables set) has NOT changed.

In general, the solution mix of an LP does NOT change when the rhs of a constraint is within its lower and upper bounds. Namely;

Keeping $b_2 = 120$ constant, we have the same solution mix for $30 \leq b_2 \leq 80$.

Keeping $b_1 = 60$ constant, we have the same solution mix for $60 \leq b_1 \leq 160$.

This is an important solution for the manager since she can know if the variables set to produce will change when a change in available sources occurs. However, there's something more important than this conclusion.

There's an alternative way of calculating New objective value without resolving the problem. Here, we will recall the definition of "shadow prices" (Dual Value).

For a max. problem, "shadow price" is defined as the Marginal Value of an additional unit of source.

Consider the shadow price of Labor, 16. This means that "An additional working hour will increase the maximum profit by 16 units". (Likewise, decrease will decrease). This is true within the bounds of rhs which is $30 \leq b_1 \leq 80$.

Then, we have,

$$\text{Old } z = 1360$$

$$\text{increase in labor} = 60 - 40 = 20$$

$$\text{increase in } z = 20 \cdot 16 = 320$$

$$\text{New } z = 1360 + 320 = 1680. \text{ It is that simple!}$$

For another example, let available clay decreased from 120 to 70. Since 70 is in the range $60 \leq b_2 \leq 120$, we can find new optimum solution like before.

$$\text{Old } z = 1360$$

$$\text{decrease in clay} = 120 - 70 = 50$$

$$\text{decrease in } z = 50 \cdot 6 = 300$$

$$\text{New } z = 1360 - 300 = \underline{\underline{1060}}$$

Finally note that, if we are to choose between additional labor and clay, our choice will be labor since its shadow price is higher. (16)

8. Irwin Textile Mills produces two types of cotton cloth—denim and corduroy. Corduroy is a heavier grade of cotton cloth and, as such, requires 7.5 pounds of raw cotton per yard, whereas denim requires 5 pounds of raw cotton per yard. A yard of corduroy requires 3.2 hours of processing time; a yard of denim requires 3.0 hours. Although the demand for denim is practically unlimited, the maximum demand for corduroy is 510 yards per month. The manufacturer has 6,500 pounds of cotton and 3,000 hours of processing time available each month. The manufacturer makes a profit of \$2.25 per yard of denim and \$3.10 per yard of corduroy. The manufacturer wants to know how many yards of each type of cloth to produce to maximize profit.
- Formulate a linear programming model for this problem.
 - Transform this model into standard form.
10. Solve the linear programming model formulated in Problem 8 for Irwin Mills by using the computer.
- If Irwin Mills can obtain additional cotton or processing time, but not both, which should it select? How much? Explain your answer.
 - Identify the sensitivity ranges for the objective function coefficients and for the constraint quantity values. Then explain the sensitivity range for the demand for corduroy.
 - How much extra cotton and processing time are left over at the optimal solution? Is the demand for corduroy met?
 - What is the effect on the optimal solution if the profit per yard of denim is increased from \$2.25 to \$3.00? What is the effect if the profit per yard of corduroy is increased from \$3.10 to \$4.00?
 - What would be the effect on the optimal solution if Irwin Mills could obtain only 6,000 pounds of cotton per month?

Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X1	456	0	2,25	0	2,9063
X2	510	0	3,1	2,4	Infinity
Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Constraint 1	0	395	6500	6105	Infinity
Constraint 2	0,75	0	3000	1632	3237
Constraint 3	0,7	0	510	0	692,3077

8) a) Maximize $Z = 2,25x_1 + 3,10x_2$

subject to (1) $5x_1 + 7,5x_2 \leq 6500$

(2) $3x_1 + 3,2x_2 \leq 3000$

$x_1, x_2 \geq 0$ (3) $x_2 \leq 510$

	Raw Cotton	Processing Time	PROFIT
(x ₁) Denim	5	3,0	2,25
(x ₂) Corduroy	7,5	3,2	3,10
AVAILABLE	6500	3000	

$x_2 \leq 510$

b) Maximize $Z = 2,5x_1 + 3,10x_2$

$$\text{subject to } 5x_1 + 7,5x_2 + s_1 = 6500$$

$$3x_1 + 3,2x_2 + s_2 = 3000$$

$$x_2 + s_3 = 510 \quad \text{(Note: } s_3 \text{ is written as } 510\text{)} \quad x_1, x_2 \geq 0$$

10) a) Processing time because its dual value is higher

b) Objective function coefficients;

$$0 \leq c_1 \leq 2,91$$

$$2,4 \leq c_2 \leq \infty$$

Constraints;

$$6105 \leq b_1 \leq \infty$$

$$1632 \leq b_2 \leq 3237$$

$$0 \leq b_3 \leq 692,31$$

The current solution mix will remain optimal for demand in the range $(0; 692)$. Moreover, within this range, an additional increase in demand will increase the profit by 0,7 units (\$)

c) 395 pounds of raw cotton left and NO processing time left.
Demand is met at its max. value 510.

d) we cannot know the Dulin effect since 3\$ is out of the sensitivity range. We should resolve the question.

$$\text{Cordury} \Rightarrow \text{New Profit} = Z = (456 \cdot 2,25 + 510 \cdot 3,1) + (4 - 3,1) \cdot 510 \\ = 2607 + 459 = 3066 \quad \text{increase}$$

e) New Profit ~~will not~~ cannot be obtained since 6000 is out of the range.

14. Rucklehouse Public Relations has been contracted to do a survey following an election primary in New Hampshire. The firm must assign interviewers to conduct the survey by telephone or in person. One person can conduct 80 telephone interviews or 40 personal interviews in a single day. The following criteria have been established by the firm to ensure a representative survey:

- At least 3,000 interviews must be conducted.
- At least 1,000 interviews must be by telephone.
- At least 800 interviews must be personal.

An interviewer conducts only one type of interview each day. The cost is \$50 per day for a telephone interviewer and \$70 per day for a personal interviewer. The firm wants to know the minimum number of interviewers to hire in order to minimize the total cost of the survey.

15. Formulate a linear programming model for this problem.
16. Solve the linear programming model formulated in Problem 14 for Rucklehouse Public Relations by using the computer.
- a. If the firm could reduce the minimum interview requirement for either telephone or personal interviews, which should the firm select? How much would a reduction of one interview in the requirement you selected reduce total cost? Solve the model again, using the computer, with the reduction of this one interview in the constraint requirement to verify your answer.
 - b. Identify the sensitivity ranges for the cost of a personal interview and the number of total interviews required.
 - c. Determine the sensitivity ranges for the daily cost of a telephone interviewer and the number of personal interviews required.
 - d. Does the firm conduct any more telephone and personal interviews than are required, and if so, how many more?
 - e. What would be the effect on the optimal solution if the firm were required by the client to increase the number of personal interviews conducted from 800 to a total of 1,200?

Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X1	27,5	0	50	0	140
X2	20	0	70	25	Infinity
Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Constraint 1	-0,625	0	3000	1800	Infinity
Constraint 2	0	1200	1000	-Infinity	2200
Constraint 3	-1,125	0	800	0	2000

14) x_1 : Number of telephone interviewers \rightarrow 80 int./day
 x_2 : Number of personal interviewers \rightarrow 40 int./day

Minimize $Z = 50x_1 + 70x_2$

subject to $80x_1 + 40x_2 \geq 3000$
 $80x_1 \geq 1000$

$$40x_2 \geq 800$$

$$x_1, x_2 \geq 0$$

16) a) Personnel interviews, because its Dual Value is less.
One interview would reduce the cost by +1,125\$.

b) Cost of a personnel interview; $25 \leq c_2 \leq \infty$

Total interviews required; $1800 \leq b_1 \leq \infty$

c) Daily cost of telephone interview; $0 \leq c_1 \leq 140$

Number of personnel interviews; $0 \leq b_3 \leq 2000$

d) 1200 more telephone interviews

e) Cost will increase by; $(1200 - 800) \cdot 1,125 = 450$

Current Cost is $z = 27,550 + 20.70 = 2775$

New Cost will be: $z = 2775 + 450 = 3225$

11. The Bradley family owns 410 acres of farmland in North Carolina on which they grow corn and tobacco. Each acre of corn costs \$105 to plant, cultivate, and harvest; each acre of tobacco costs \$210. The Bradleys have a budget of \$52,500 for next year. The government limits the number of acres of tobacco that can be planted to 100. The profit from each acre of corn is \$300; the profit from each acre of tobacco is \$520. The Bradleys want to know how many acres of each crop to plant in order to maximize their profit.

Formulate a linear programming model for this problem.

13. Solve the linear programming model formulated in Problem 11 for the Bradley farm by using the computer.

a. The Bradleys have an opportunity to lease some extra land from a neighbor. The neighbor is offering the land to them for \$110 per acre. Should the Bradleys lease the land at that price? What is the maximum price the Bradleys should pay their neighbor for the land, and how much land should they lease at that price?

b. The Bradleys are considering taking out a loan to increase their budget. For each dollar they borrow, how much additional profit would they make? If they borrowed an additional \$1,000, would the number of acres of corn and tobacco they plant change?

c. How many acres of farmland will not be cultivated at the optimal solution? Do the Bradleys use the entire 100-acre tobacco allotment?

d. What would the profit for corn have to be for the Bradleys to plant only corn?

e. If the Bradleys can obtain an additional 100 acres of land, will the number of acres of corn and tobacco they plan to grow change?

f. If the Bradleys decide not to cultivate a 50-acre section as part of a crop recovery program, how will it affect their crop plans?

Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X1	320	0	300	260	520
X2	90	0	520	300	600
Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Constraint 1	80	0	410	400	500
Constraint 2	2,0952	0	52500	43050	53550
Constraint 3	0	10	100	90	Infinity

11) x_1 : Acre of corn grown

x_2 : Acre of tobacco grown

$$\text{Maximize } z = 300x_1 + 520x_2$$

$$\text{subject to } x_1 + x_2 \leq 410$$

$$105x_1 + 210x_2 \leq 52500$$

$$x_1, x_2 \geq 0 \quad x_2 \leq 100$$

13) a) No, because an additional acre has extra profit of 80\$. They should pay less than 80\$.

b) $52500 + 1000 = 53500$ is in the range $(43050; 53550)$. For each dollar they borrow, they increase their profit by 2,1\$ and total $= 2,1 \cdot 1000 = 2100$ \$. The optimum values for x_1 and x_2 will then increase.

c) All land is cultivated. 10 acre tobacco still available.

d) At least 520\$

e) Yes, because Dual value of First constraint > 0

f) $410 - 50 = 360$ NOT in range $(400; 500)$. Resolution of problem is required to answer this.

36. Marie McCoy has committed to the local PTA to make some items for a bake sale on Saturday. She has decided to make some combination of chocolate cakes, loaves of white bread, custard pies, and sugar cookies. Thursday evening she went to the store and purchased 20 pounds of flour, 10 pounds of sugar, and 3 dozen eggs, which are the three main ingredients in all the baked goods she is thinking about making. The following table shows how much of each of the main ingredients is required for each baked good:

	Ingredient			
	Flour (cups)	Sugar (cups)	Eggs	Baking Time (min.)
Cake	2.5	2	2	45
Bread	9	0.25	0	35
Pie	1.3	1	5	50
Cookies	2.5	1	2	16

There are 18.5 cups in a 5 pound bag of flour and 12 cups in a 5 pound bag of sugar. Marie plans to get up and start baking on Friday morning after her kids leave for school and finish before they return after soccer practice (8 hours). She knows that the PTA will sell a chocolate cake for \$12, a loaf of bread for \$8, a custard pie for \$10, and a batch of cookies for \$6. Marie wants to decide how many of each type of baked good she should make in order for the PTA to make the most money possible.

Formulate a linear programming model for this problem.

37. Solve the linear programming model formulated in Problem 36 for Marie McCoy.
- Are any of the ingredients left over?
 - If Marie could get more of any ingredient, which should it be? Why?
 - If Marie could get 6 more eggs, 20 more cups of flour, or 30 more minutes of oven time, which should she choose? Why?

Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X1	2,7318	0	12	6	12,6286
X2	3,2222	0	8	7,2414	16,9241
X3	0	6,6517	10	-infinity	16,6517
X4	15,2682	0	6	5,3714	12
Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Constraint 1	0,0843	0	74	50,8378	94,3714
Constraint 2	0	2,4626	24	21,5374	infinity
Constraint 3	1,2395	0	36	10,8976	42,8974
Constraint 4	0,2069	0	480	400,7778	551,4167

36)		Flour	Sugar	Eggs	Time (min)	PROFIT
	(x ₁) Cake	2,5	2	2	45	12
	(x ₂) Bread	9	0,25	0	35	8
	(x ₃) Pie	1,3	1	5	50	10
	(x ₄) Cookies	2,5	1	2	16	6
	AVAILABLE	18,5-4 =74	12,2 =24	12,3 =36	8,60 =480	

36) Maximize $Z = 12x_1 + 8x_2 + 10x_3 + 6x_4$

subject to $2.5x_1 + 9x_2 + 1.3x_3 + 2.5x_4 \leq 74$

$$2x_1 + 0.25x_2 + x_3 + x_4 \leq 24$$

$$2x_1 + 5x_3 + 2x_4 \leq 36$$

$$45x_1 + 35x_2 + 50x_3 + 16x_4 \leq 480$$

$$x_1, x_2, x_3, x_4 \geq 0$$

37) a) Yes, 2.46 cups of sugar left over

b) Eggs, because it has the most Dual Value

c) 6 more eggs: $1.24 \cdot 6 = \underline{\underline{7.44}} \rightarrow \text{max. Choose}$
 6 more eggs.

20 more cups of flour: $0.084 \cdot 20 = 1.68$

30 more minutes: $0.207 \cdot 30 = 6.21$

A final Note:

Excel GM gives the bound as, lets say

#	Allowable Increase	Allowable Decrease
x_1	6	6,6286
x_2	0,7586	8,9241
x_3	+infinity	6,6517
x_4	0,6286	6

Be careful! These values are absolute differences
 of /Bound - Original Value/