

## STOCHASTIC MODELS LECTURE NOTES

CHAPTERS

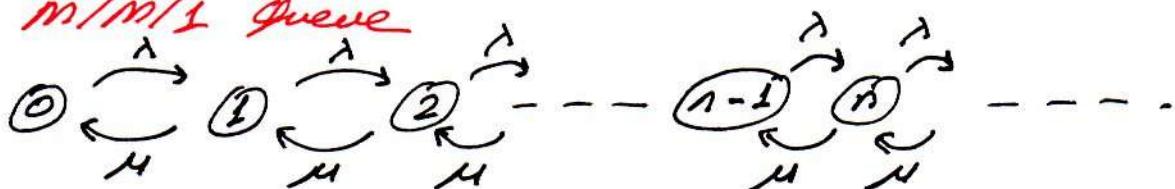
17.6 - 17.7 - 17.9

### M/M Models

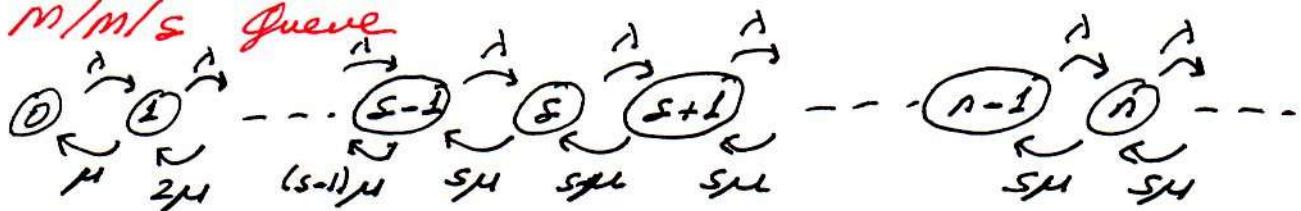
#### \* M/M/s Models

- (i) All interarrival times are  $\sim \text{i.i.d Exponential}(\lambda)$  constant
  - (ii) All service times are  $\sim \text{i.i.d Exponential}(\mu)$  constant
  - (iii) The # of servers is  $s$ ; no restriction on capacity or calling population
- Rate Diagrams:

#### M/M/1 Queue



#### M/M/s Queue



#### Notes

- (i) We reach steady state when  $\rho = \frac{\lambda}{s \cdot \mu} < 1$
- (ii) We find  $C_n$ ,  $P_0$  and  $P_n$  as before: "Birth & Death process." We have;  $L = \sum_{n=0}^{\infty} n \cdot P_n$  and we find other performance measures using Little's result. If we know the model, we readily have the formulas (we do NOT need to calculate from basic definitions.)

*Results for M/M/1 case;*

$$C_n = \frac{\lambda_{n+1} \cdot \lambda_{n+2} \cdot \dots \cdot \lambda_0}{\mu_n \cdot \mu_{n+1} \cdot \dots \cdot \mu_1} = \left(\frac{\lambda}{\mu}\right)^n = \rho^n$$

$$P_0 = \left( \sum_{n=0}^{\infty} C_n \right)^{-1} = \left( \sum \rho^n \right)^{-1} = \left( \frac{1}{1-\rho} \right)^{-1} = 1-\rho$$

$$P_n = C_n \cdot P_0 = \rho^n \cdot (1-\rho)$$

$$L = \sum_{n=0}^{\infty} n \cdot P_n = \sum_{n=0}^{\infty} n \cdot \rho^n \cdot (1-\rho) = \dots = \frac{\lambda}{\mu - \lambda}$$

$$W = \frac{L}{\lambda} = \frac{1}{\mu - \lambda}; W_q = W - \frac{1}{\mu} = \frac{1}{\mu(\mu - \lambda)}; L_q = W_q \cdot \lambda = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

*Additionally;*

$$\begin{aligned} P(N > t) &= e^{-\mu(1-\rho) \cdot t} \\ P(N_q > t) &= \rho \cdot e^{-\mu(1-\rho) \cdot t} \end{aligned} \quad \left. \begin{array}{l} \text{Use these formula} \\ \text{to find probabilities} \\ \text{about waiting time.} \end{array} \right\}$$

\* Remember;  $N$ : Random variable, waiting time of the customer in the system, whose average is  $W = E(N)$

17.6-5. It is necessary to determine how much in-process storage space to allocate to a particular work center in a new factory. Jobs arrive at this work center according to a Poisson process with a mean rate of 3 per hour, and the time required to perform the necessary work has an exponential distribution with a mean of 0.25 hour. Whenever the waiting jobs require more in-process storage space than has been allocated, the excess jobs are stored temporarily in a less convenient location. If each job requires 1 square foot of floor space while it is in in-process storage at the work center, how much space must be provided to accommodate all waiting jobs (a) 50 percent of the time, (b) 90 percent of the time, and (c) 99 percent of the time? Derive an analytical expression to answer these three questions. Hint: The sum of a geometric series is

$$\sum_{n=0}^N x^n = \frac{1 - x^{N+1}}{1 - x}.$$

$$\sum_{n=0}^N P_n = \sum_{n=0}^N 0,75^n \cdot 0,25 = 0,25 \cdot \frac{1 - 0,75^{N+1}}{1 - 0,75} = 1 - 0,75^{N+1} = 0,90$$

17.6-5) Consider M/M/1 model.

$$\rho = \frac{3}{4} = 0,75; P_0 = 1 - \rho = 0,25$$

$$P_n = \rho^n \cdot (1-\rho) = 0,75^n \cdot 0,25$$

$$\text{b)} \text{ we want, } f(N) = \sum_{n=0}^N P_n > 0,90$$

Note that,  $f(N)$  is an increasing function.  
Let  $f(N) = 0,90$

$$\text{Then, } 0,75^{N+1} = 0,10$$

$$(N+1) \cdot \ln 0,75 = \ln 0,10 \Rightarrow N+1 = \frac{\ln 0,10}{\ln 0,75} = 8,004$$

$$N = 7,004$$

$N$  should be at least 8.

In general, let we want this  $100q\%$  of the time.

Then,  $N$  should be at least  $\left\lceil \frac{1}{0,75} \cdot \ln(1-q) \right\rceil$  Round UP function!

17.6-9. The Friendly Neighbor Grocery Store has a single checkout stand with a full-time cashier. Customers arrive randomly at the stand at a mean rate of 30 per hour. The service-time distribution is exponential, with a mean of 1.5 minutes. This situation has resulted in occasional long lines and complaints from customers. Therefore, because there is no room for a second checkout stand, the manager is considering the alternative of hiring another person to help the cashier by bagging the groceries. This help would reduce the expected time required to process a customer to 1 minute, but the distribution still would be exponential.

The manager would like to have the percentage of time that there are more than two customers at the checkout stand down below 25 percent. She also would like to have no more than 5 percent of the customers needing to wait at least 5 minutes before beginning service, or at least 7 minutes before finishing service.

- (a) Use the formulas for the M/M/1 model to calculate  $L$ ,  $W$ ,  $W_q$ ,  $L_q$ ,  $P_0$ ,  $P_1$ , and  $P_2$  for the current mode of operation. What is the probability of having more than two customers at the checkout stand?
- T (b) Use the Excel template for this model to check your answers in part (a). Also find the probability that the waiting time before beginning service exceeds 5 minutes, and the probability that the waiting time before finishing service exceeds 7 minutes.
- (c) Repeat part (a) for the alternative being considered by the manager.
- (d) Repeat part (b) for this alternative.
- (e) Which approach should the manager use to satisfy her criteria as closely as possible?

17.6-9)  $\lambda = 30 \text{ customers/hour}$ , Exponential arrive randomly

$$\mu = \frac{1}{1,5} \cdot 60 = 40 \text{ customers/hour}$$

Single checkout  $\Rightarrow \delta = 1$

Hire another person  $\Rightarrow \mu_{new} = 60 \text{ cust/hour}$

(still  $\delta = 1$  and exponential)

The Manager's Desires;

$$(i) P(N > 2) < 0,25$$

$$(ii) P(wq \geq \frac{5}{60}) < 0,05$$

$$(iii) P(w \geq \frac{7}{60}) < 0,05$$

↪ Note that, these are hours

M/M/1 model

$$a) \rho = \frac{\lambda}{\mu} = \frac{30}{60} = 0,75$$

$$P_0 = 1 - \rho = 0,25 ; P_1 = \rho \cdot (1-\rho) = 0,75 \cdot 0,25 = 0,188$$

$$P_2 = \rho^2 \cdot (1-\rho) = 0,75^2 \cdot 0,25 = 0,141$$

$$P(N > 2) = 1 - P(N \leq 2) = 1 - [P_0 + P_1 + P_2] = 0,421 > 0,25$$

(i) is NOT satisfied!

$$L = \frac{\lambda}{\mu - \lambda} = \frac{30}{40-30} = 3 ; \quad L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{30^2}{40(60-30)} = 2,25$$

$$W = \frac{1}{\mu - \lambda} = \frac{1}{10} \text{ hours} = 6 \text{ min}; \quad W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{30}{40(60-30)} = \frac{3}{65} \text{ hours} = 4,5 \text{ min}$$

b)  $P(W_q > \frac{5}{60}) = 0,75 \cdot e^{-60 \cdot (1-0,125) \cdot \frac{5}{60}} = 0,326 > 0,05$

(ii) is NOT satisfied!

$$P(W_q > \frac{7}{60}) = e^{-60 \cdot (1-0,125) \cdot \frac{7}{60}} = 0,311 > 0,05$$

(iii) is NOT satisfied!

c)  $\mu_{new} = 60; \lambda = 30; M/M/1$  model

$$\rho = \frac{30}{60} = 0,50 \Rightarrow P_0 = 1 - 0,50 = 0,50; \quad P_1 = 0,50 \cdot 0,50 = 0,25 \\ P_2 = 0,50^2 = 0,25 = 0,125$$

$$P(N > 2) = 1 - [0,50 + 0,25 + 0,125] = 0,125 < 0,25$$

(i) is satisfied!

$$L = \frac{30}{60-30} = 1; \quad L_q = \frac{30^2}{60(60-30)} = 0,5$$

$$W = \frac{1}{60-30} = \frac{1}{30} \text{ hours} = 2 \text{ min}; \quad W_q = \frac{30}{60(60-30)} = \frac{1}{60} \text{ hours} = 1 \text{ min.}$$

d)  $P(W_q > \frac{5}{60}) = 0,50 \cdot e^{-60 \cdot (1-0,50) \cdot \frac{5}{60}} = 0,041 < 0,05$

(ii) is satisfied

$$P(W > \frac{7}{60}) = e^{-60 \cdot (1-0,50) \cdot \frac{7}{60}} = 0,03 < 0,05$$

(iii) is satisfied

e) So, everything is just as the way she wants, when she takes the new employee. She should absolutely take him :)

\* Results for the M/M/s case;

$$c_n = \begin{cases} \frac{(\lambda/\mu)^n}{n!} & \text{for } n=1, 2, \dots, s-1 \\ \frac{(\lambda/\mu)^n}{s! \cdot s^{n-s}} & \text{for } n=s, s+1, s+2, \dots \end{cases}$$

$$P_0 = \left[ \sum_{n=0}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!} \cdot \frac{1}{1 - \frac{\lambda}{s\mu}} \right]^{-1}$$

$$P_n = \begin{cases} \frac{(\lambda/\mu)^n}{n!} \cdot P_0 & \text{for } 0 \leq n < s \\ \frac{(\lambda/\mu)^n}{s! \cdot s^{n-s}} \cdot P_0 & \text{for } n \geq s \end{cases}$$

$$L_q = P_0 \cdot \frac{(\lambda/\mu)^s \cdot \rho}{s! \cdot (1-\rho)^2} \quad \text{where } \rho = \frac{\lambda}{s\mu}$$

$$P(N>t) = e^{-\mu t} \left[ \frac{1 + P_0 (\lambda/\mu)^s}{s! (1-\rho)} \left( \frac{1 - e^{-\mu t(s-1-\lambda/\mu)}}{s-1-\lambda/\mu} \right) \right]$$

$$P(W_q>t) = (1 - P\{W_q=0\}) \cdot e^{-\mu(1-\rho)t}$$

where  $P\{W_q=0\} = \sum_{n=0}^{s-1} P_n$   
(prob. that new arriving customer NOT wait in the queue)

- 17.6-14. A gas station with only one gas pump employs the following policy: If a customer has to wait, the price is \$1 per gallon; if she does not have to wait, the price is \$1.20 per gallon. Customers arrive according to a Poisson process with a mean rate of 15 per hour. Service times at the pump have an exponential distribution with a mean of 3 minutes. Arriving customers always wait until they can eventually buy gasoline. Determine the expected price of gasoline per gallon.

$$P(Y=1) = P(N>1/N>0) = 1 - 0.75 = 0.25$$

$$\begin{aligned} P(Y=1.2) &= P(N=1/N>0) = \frac{P(N=1)}{P(N>0)} = \frac{P(1-\rho)}{1-\rho} \\ &= 0.75 \end{aligned}$$

(52)

$$E(y) = \sum y \cdot p_y = 1,20 \cdot 0,25 + 1 \cdot 0,75 = 1,05$$

T 17.6-11. The Security & Trust Bank employs 4 tellers to serve its customers. Customers arrive according to a Poisson process at a mean rate of 2 per minute. However, business is growing and management projects that the mean arrival rate will be 3 per minute a year from now. The transaction time between the teller and customer has an exponential distribution with a mean of 1 minute.

Management has established the following guidelines for a satisfactory level of service to customers. The average number of customers waiting in line to begin service should not exceed 1. At least 95 percent of the time, the number of customers waiting in line should not exceed 5. For at least 95 percent of the customers, the time spent in line waiting to begin service should not exceed 5 minutes.

- (a) Use the  $M/M/s$  model to determine how well these guidelines are currently being satisfied.
- (b) Evaluate how well the guidelines will be satisfied a year from now if no change is made in the number of tellers.
- (c) Determine how many tellers will be needed a year from now to completely satisfy these guidelines.

17.6-11)  $s = 4$  servers

$$\lambda = 2 \cdot 60 = 120 \text{ customers/hour}$$

$$\lambda_{\text{next}} = 3 \cdot 60 = 180 \text{ customers/hour}$$

$$\mu = \frac{60}{1} = 60 \text{ customers/hour}$$

$M/M/4$  Model

Manager Wants:

$$(i) L_q \leq 1$$

$$(ii) P(N_q \leq 5) \geq 0,95$$

$$(iii) P(N_q \leq \frac{5}{60}) \geq 0,95$$

$$a) P_0 = \left[ 1 + \sum_{n=1}^3 \frac{(120/60)^n}{n!} + \frac{(120/60)^4}{4!} \cdot \sum_{n=4}^{\infty} \left( \frac{120}{4 \cdot 60} \right)^{n-4} \right]^{-1}$$

$$= \left[ 1 + 2 + \frac{2^2}{2} + \frac{2^3}{6} + \frac{2^4}{24} \cdot \frac{1}{1 - \frac{120}{240}} \right]^{-1} = 0,1304$$

$$P_n = \frac{(120/60)^n}{n!} \cdot P_0 \quad \text{for } 0 \leq n \leq 4$$

$$P_1 = 2P_0 = 0,2608$$

$$P_2 = 2P_0 = 0,2608$$

$$P_3 = \frac{4}{3} P_0 = 0,1793$$

$$P_n = \frac{(120/60)^n}{4! \cdot 4^{n-4}} P_0 \quad \text{for } n \geq 4$$

$$P_4 = \frac{2^4}{4! \cdot 6^0} \cdot P_0 = \frac{16}{24} P_0 = 0,0869$$

$$L_q = \frac{0,1304 \cdot (120/60)^4 \cdot \left( \frac{120}{4 \cdot 60} \right)}{4! \cdot \left( 1 - \frac{120}{4 \cdot 60} \right)^2} = 0,174 \quad \text{1. (i) is Satisfied}$$

$$P_5 = \frac{2^5}{4! \cdot 6^1} P_0 = \frac{16}{68} P_0 = 0,0435$$

$$P_6 = \frac{2^6}{4! \cdot 6^2} P_0 = \frac{16}{96} P_0 = 0,0217$$

$$P_7 = \frac{2^7}{4! \cdot 6^3} P_0 = \frac{8}{96} P_0 = 0,0109$$

$$P_8 = \frac{2^8}{4! \cdot 6^4} P_0 = \frac{4}{96} P_0 = 0,0054$$

$$P_9 = \frac{2^9}{4! \cdot 6^5} P_0 = \frac{2}{96} P_0 = 0,0027$$

$$P(N_q \leq 5) = P(N \leq 9) = \sum_{n=0}^9 P_n = 0,1306 + \dots + 0,0027 = 0,997 > 0,95$$

(ii) is satisfied

$$P(M_q > \frac{5}{60}) = [1 - (P_0 + P_1 + P_2 + P_3)] \cdot e^{-\frac{\mu \cdot \lambda_{next}}{\mu} \cdot \frac{5}{60}} \\ = 0,8259 \cdot e^{-10} = 0,0000375$$

$$P(M_q \leq \frac{5}{60}) = 1 - 0,0000375 = 0,9999625 > 0,95$$

(iii) is satisfied

Everything is good as Manager Wants

b)  $\lambda_{next} = 180; \frac{\lambda_{next}}{\mu} = \frac{180}{60} = 3; \rho = \frac{180}{4 \cdot 60} = 0,75$

$$P_0 = \left[ 1 + 3 + \frac{3^2}{2} + \frac{3^3}{6} + \frac{3^4}{4!} \cdot \frac{1}{1 - \frac{180}{240}} \right]^{-1} = 0,0337$$

$n$	0	1	2	3	4	5	6	7	8	9
$P_n$	0,0337	0,1011	0,1517	0,1517	0,1137	0,055	0,064	0,048	0,036	0,027

$$L_q = \frac{0,0337 \cdot 3^4 \cdot 0,75}{4! \cdot (1 - 0,75)^2} = 1,365 > 1 \text{ (i) is NOT satisfied}$$

$$P(N_q \leq 5) = \sum_{n=0}^9 P_n = 0,8122 < 0,95 \text{ (ii) is NOT satisfied}$$

$$P(M_q \leq \frac{5}{60}) = 1 - 0,5618 \cdot e^{-5} = 0,9961 > 0,95 \text{ (iii) is satisfied}$$

c) If we hire 1 server; M/M/1 model

$$\frac{\lambda_{next}}{\mu} = 3; \rho = \frac{180}{5 \cdot 60} = 0,60$$

$$P_0 = \left[ 1 + 3 + \frac{3^2}{2} + \frac{3^3}{6} + \frac{3^4}{24} + \frac{3^5}{120} \cdot \frac{1}{1 - 0,60} \right]^{-1} = 0,0466$$

$$P_n = \frac{3^n}{n!} \cdot P_0 \text{ for } 0 \leq n < 5 ; \quad P_n = \frac{3^n}{5! \cdot 5^{n-5}} \text{ for } n \geq 5$$

$n$	0	1	2	3	4	5	6	7	8	9	10
$P_n$	0,0466	0,1398	0,2097	0,2097	0,1573	0,0964	0,0566	0,0360	0,0204	0,0122	0,0017

$$L_q = \frac{0,0466 \cdot 3^5 \cdot 0,60}{5! \cdot 0,40^2} = 0,354 \text{ (i) is satisfied}$$

$$P(N_q \leq 5) = P(N \leq 10) = \sum_{n=0}^{10} P_n = 0,9824 > 0,95 \text{ (ii) is satisfied}$$

$$P\left\{N_q \leq \frac{5}{60}\right\} = 1 - 0,2369 \cdot e^{-10} = 0,99998924 > 0,95 \text{ (iii) is satisfied}$$

So, for the new case, some conditions would NOT be satisfied for  $s=4$ . Hiring a new server;  $s=5$  case, satisfies all 3 conditions.

\* Results for the  $M/M/\infty$  case;

Here, customers are also servers. So, there's no queue

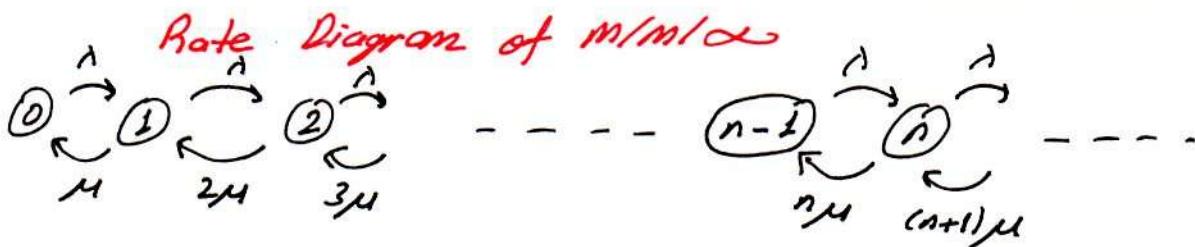
$$C_n = \frac{(\lambda/\mu)^n}{n!} \text{ for } n=0, 1, \dots$$

$$P_0 = \left( \sum_{n=0}^{\infty} C_n \right)^{-1} = \left[ \sum_{n=0}^{\infty} \frac{(\lambda/\mu)^n}{n!} \right]^{-1} = e^{-\lambda/\mu} ; \quad P_n = \frac{(\lambda/\mu)^n}{n!} \cdot e^{-\lambda/\mu}$$

$$L = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n \cdot \frac{(\lambda/\mu)^n}{n!} \cdot e^{-\lambda/\mu} = \frac{\lambda}{\mu} \cdot \underbrace{\sum_{n=1}^{\infty} e^{-\lambda/\mu} \cdot \frac{(\lambda/\mu)^{n-1}}{(n-1)!}}_{= \sum_{n=0}^{\infty} P_n = 1} = \frac{\lambda}{\mu}$$

$\rho = 0$  so, steady state always exists

$$W = \frac{L}{\lambda} = \frac{1}{\mu} : \text{Expected service time}; \quad W_q = 0$$



Examples;

- (i) Self service (assume unlimited food)
- (ii) Seaside
- (iii) Internet

\* Results for  $M/M/1/K$

for  $\rho < 1$ ;

$$c_n = \left(\frac{\lambda}{\mu}\right)^n = \rho^n \text{ for } n = 0, 1, \dots, K$$

$$P_0 = \frac{1-\rho}{1-\rho^{K+1}} ; P_n = \frac{1-\rho}{1-\rho^{K+1}} \cdot \rho^n \text{ for } n = 0, 1, \dots, K$$

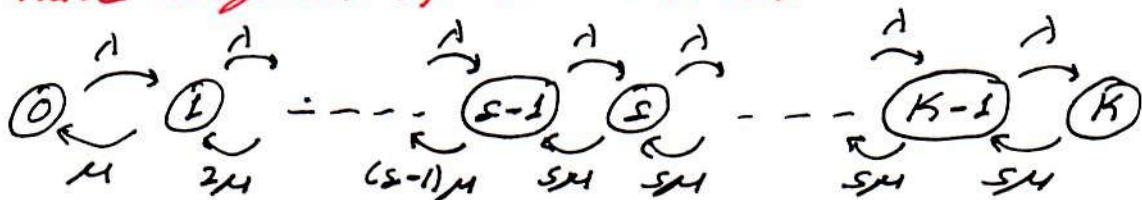
$$L = \frac{\rho}{1-\rho} - \frac{(K+1) \cdot \rho^{K+1}}{1-\rho^{K+1}}$$

$$L_q = L - (1 - P_0) ; W = \frac{L}{\lambda} ; W_q = \frac{L_q}{\lambda} ; \bar{\lambda} = \lambda \cdot (1 - P_K)$$

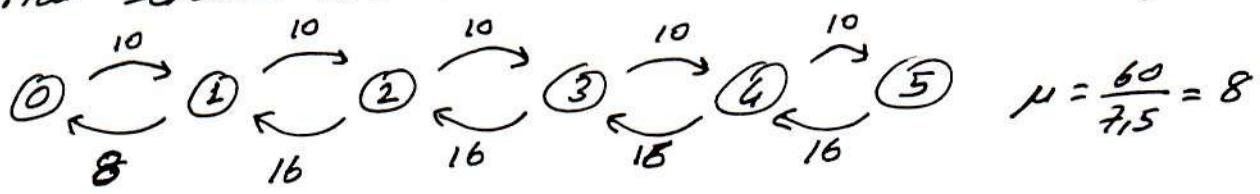
for  $\rho = 1$ ; (only for this case, we have steady state distribution)

$$P_n = \frac{1}{K+1} \text{ for } n = 0, 1, \dots, K \text{ AND } L = \frac{K}{2}$$

Rate diagram of  $M/M/s/K$



**Example** Consider a Barber Shop with 2 servers and 3 waiting chairs. There are also newspapers for waiting customers, which is irrelevant to the subject. Anyway, let 10 customers/hour comes and average time to cut a hair is 7,5 minutes. They are too fast, right? Assume Exponential service and interarrival times. we have;



$$P_1 = \frac{10}{8} P_0$$

$$\sum_{n=0}^{\infty} P_n = 1$$

$$P_2 = \frac{10^2}{8 \cdot 16} P_0$$

$$P_0 = \left[ 1 + \frac{10}{8} + \frac{10^2}{8 \cdot 16} + \dots + \frac{10^5}{8 \cdot 16^4} \right]^{-1}$$

$$\vdots$$

--- etc.

$$P_5 = \frac{10^5}{8 \cdot 16^4} P_0$$

17.6-25. Janet is planning to open a small car-wash operation, and she must decide how much space to provide for waiting cars.

Janet estimates that customers would arrive randomly (i.e., a Poisson input process) with a mean rate of 1 every 4 minutes, unless the waiting area is full, in which case the arriving customers would take their cars elsewhere. The time that can be attributed to washing one car has an exponential distribution with a mean of 3 minutes. Compare the expected fraction of potential customers that will be lost because of inadequate waiting space if (a) 0 spaces (not including the car being washed), (b) 2 spaces, and (c) 4 spaces were provided.

$$17.6-25) \lambda = \frac{60}{4} = 15 \text{ cars/hour}$$

$$\mu = \frac{60}{3} = 20 \text{ cars/hour}$$

m/m/1/K model  $K = 1; 3; 5$

$$a) K = 1; \rho = \frac{15}{20} = 0,75$$

$$P_0 = \frac{1 - 0,75}{1 - 0,75^2} = 0,5714$$

$$P(\text{lost customer}) = P_1 = 0,5714 \cdot 0,75 = 0,4286$$

$$b) K = 3$$

$$P_0 = \frac{1 - 0,75}{1 - 0,75^3} = 0,3657$$

$$P_3 = 0,3657 \cdot 0,75^3 = 0,1563$$

$$c) K = 5$$

$$P_0 = \frac{1 - 0,75}{1 - 0,75^5} = 0,3041 ; P_5 = 0,3041 \cdot 0,75^5 = 0,0722$$

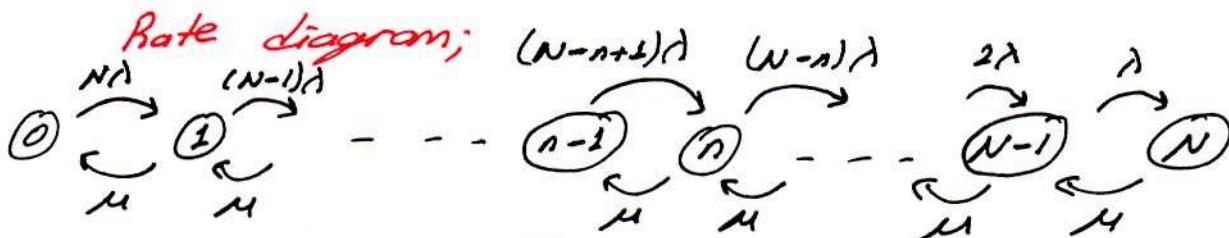
\*Results for  $m/m/1/\infty/N$

$$C_n = \frac{N!}{(N-n)!} \cdot \left(\frac{\lambda}{\mu}\right)^n \text{ for } n=0, 1, \dots, N \quad \begin{matrix} \rightarrow \text{Max. # of} \\ \text{calling population} \end{matrix}$$

$$P_0 = \left( \sum_{n=0}^N \left[ \frac{N!}{(N-n)!} \cdot \left(\frac{\lambda}{\mu}\right)^n \right] \right)^{-1}; P_n = \frac{N!}{(N-n)!} \cdot \left(\frac{\lambda}{\mu}\right)^n \cdot P_0$$

$$L_q = N - \frac{\lambda + \mu}{\lambda} (L - P_0); L = L_q + (1 - P_0); \bar{\lambda} = \lambda \cdot (N - L)$$

Note that  $\lambda$  is given for "each customer" here. A usual question for this model is machine repair.



17.6-29. At the Forrester Manufacturing Company, one repair technician has been assigned the responsibility of maintaining three machines.

For each machine, the probability distribution of the running time before a breakdown is exponential, with a mean of 9 hours. The repair time also has an exponential distribution, with a mean of 2 hours.

(a) Which queueing model fits this queueing system?

T (b) Use this queueing model to find the probability distribution of the number of machines not running, and the mean of this distribution.

(c) Use this mean to calculate the expected time between a machine breakdown and the completion of the repair of that machine.

(d) What is the expected fraction of time that the repair technician will be busy?

T (e) As a crude approximation, assume that the calling population is infinite and that machine breakdowns occur randomly at a mean rate of 3 every 9 hours. Compare the result from part (b) with that obtained by making this approximation while using (i) the  $M/M/s$  model and (ii) the finite queue variation of the  $M/M/s$  model with  $K = 3$ .

T (f) Repeat part (b) when a second repair technician is made available to repair a second machine whenever more than one of these three machines require repair.

17.6-29)  $\lambda = \frac{1}{9}$  machines / hour

$$\mu = \frac{1}{2}$$
 machines / hour

$$N = 3 \text{ machines}; \sigma = 1$$

a)  $m/m/1 \cdot /3$  Model

$$b) P_0 = \left[ 1 + \frac{3!}{2!} \left(\frac{2}{9}\right) + \frac{3!}{1!} \left(\frac{2}{9}\right)^2 + \frac{3!}{0!} \left(\frac{2}{9}\right)^3 \right]^{-1} = 0,6929$$

$$P_1 = \frac{3!}{2!} \left(\frac{2}{9}\right)^1 \cdot 0,6929 = 0,3286$$

$$P_2 = \frac{3!}{1!} \cdot \left(\frac{2}{9}\right)^2 \cdot 0,6929 = 0,1460$$

$$P_3 = \frac{3!}{0!} \cdot \left(\frac{2}{9}\right)^3 \cdot 0,6929 = 0,0325$$

N	0	1	2	3
$P_n$	0,6929	0,3286	0,1460	0,0325

$$L_q = 3 - \frac{\frac{1}{9} + \frac{1}{2}}{\frac{1}{9}} \cdot (1 - 0,6929) = 3 - \frac{11}{2} \cdot 0,5071 = 0,211$$

$$L = E(N) = 0,211 + (1 - 0,6929) = 0,718 = \sum_{n=0}^3 n \cdot P_n$$

c)  $W = ? \quad \bar{\lambda} = \frac{1}{9} \cdot (3 - 0,718) = 0,2535$

$$W = \frac{0,718}{0,2535} = 2,83 \text{ hours}$$

d)  $P(\text{Tech. is busy}) = 1 - P_0 = 1 - 0,6929 = 0,5071$

e)  $\lambda = \frac{3}{9} = \frac{1}{3}; \mu = \frac{1}{2}; \rho = \frac{\lambda}{\mu} = \frac{2}{3}$

(i) M/M/1 model

$$P_0 = 1 - \rho = \frac{1}{3} = 0,3333; P_n = \rho^n \cdot P_0 = \frac{1}{3} \rho^n$$

$$P_1 = 0,1111; P_2 = 0,0741; P_3 = 0,0494; P(N > 3) = 1 - P(N \leq 3) = 0,1976$$

$$L = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{3}}{\frac{1}{2} - \frac{1}{3}} = \frac{6}{3} = 2, L \text{ is higher because for } M/M/1/1/3 \text{ case; } P(N > 3) = 0$$

(ii) M/M/1/3 model

$$P_0 = \frac{1 - \frac{2/3}{3}}{1 - (\frac{2/3}{3})^4} = 0,4154$$

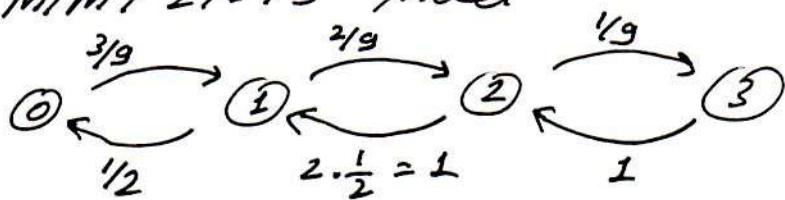
$$P_1 = 0,4154 \cdot \frac{2}{3} = 0,2769; P_2 = 0,1845; P_3 = 0,1231$$

$$L = \frac{(2/3)}{1 - (2/3)} - \frac{4 \cdot (2/3)^4}{1 - (2/3)^4} = 1,02$$

$L$  is between that of M/M/1/1/N and M/M/1.

The reason is  $P_2$  and  $P_3$  are higher this time since calling population is NOT limited.

f) M/M/2/1-13 Model



$$P_1 = \frac{3/9}{1/2} P_0 = \frac{6}{9} P_0 (= 0,364)$$

$$P_0 + P_1 + P_2 + P_3 = 1$$

$$P_2 = \frac{\frac{3}{9} \cdot \frac{2}{9}}{\frac{1}{2} \cdot 1} P_0 = \frac{12}{81} P_0 (= 0,081)$$

$$P_0 \left[ 1 + \frac{6}{9} + \frac{12}{81} + \frac{12}{729} \right] = 1$$

$$\boxed{P_0 = 0,546}$$

$$P_3 = \frac{\frac{3}{9} \cdot \frac{2}{9} \cdot \frac{1}{9}}{\frac{1}{2} \cdot 1 \cdot 1} P_0 = \frac{12}{729} P_0 (= 0,009)$$

$N$	0	1	2	3
$P_n$	0,546	0,364	0,081	0,009

$$L = 0 \cdot 0,546 + 1 \cdot 0,364 + 2 \cdot 0,081 + 3 \cdot 0,009 = 0,553$$

Two servers make the system more quick than one server, as might be expected. Also note that, part (b) can also be solved by basic "Birth and Death process"

17.6-32)

- a)  $\{X(t)\}$ : # and status of the machines that are not working at time  $t \geq 0\}$

$$SS = \{0, 1U, 1, 2U, 2, 3U, 3\}$$

where U stands for "First operation is unsuccessful."

$$\mu = 2 ; \lambda = \frac{1}{3}$$

$$P(\text{Perform op. second time}) = \frac{1}{3}$$

- (a) How should the states of the system be defined in order to formulate this queueing system as a continuous time Markov chain? (Hint: Given that a first operation is being performed on a failed machine, completing this operation *successfully* and completing it *unsuccessfully* are two separate events of interest. Then use Property 6 regarding disaggregation for the exponential distribution.)
- (b) Construct the corresponding rate diagram.
- (c) Develop the balance equations.

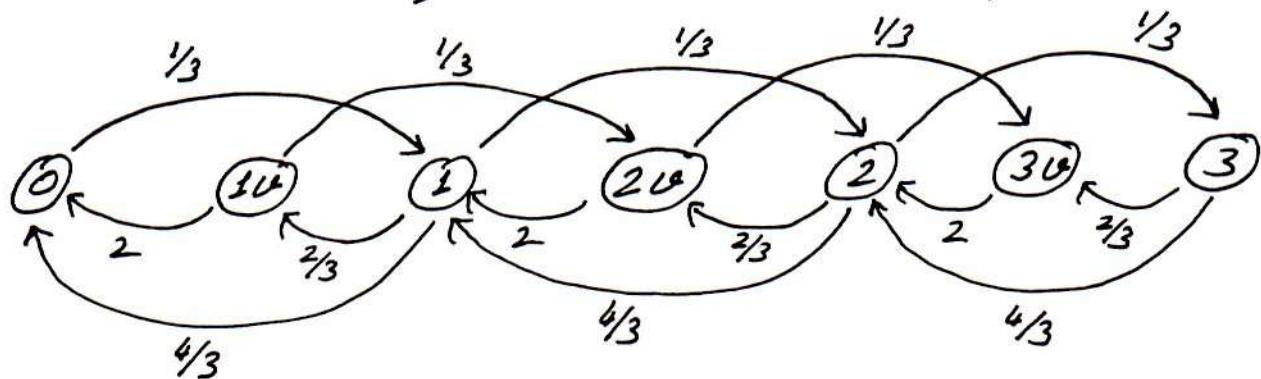
b) let  $T$  is time for operation.  $T \sim \text{Exponential}(\mu=2)$

let  $T_u$  is successful operation and  $T_{su}$  is unsuccessful operation. By disaggregation property;

$$T \xrightarrow{P_u = \frac{1}{3}} T_u \sim \text{Exponential}(\mu_u = \frac{1}{3} \cdot 2 = \frac{2}{3})$$

$$T \xrightarrow{P_s = \frac{2}{3}} T_{su} \sim \text{Exponential}(\mu_s = \frac{2}{3} \cdot 2 = \frac{4}{3})$$

Then, the rate diagram is as follows;



$$c) \frac{1}{3} P_0 = 2 P_{2u} + \frac{4}{3} P_1$$

$$P_0 + P_{1u} + P_1 + P_{2u} + P_2 + P_{3u} + P_3 = 1$$

$$(2 + \frac{1}{3}) P_{2u} = \frac{2}{3} P_1$$

$$\underbrace{\left( \frac{4}{3} + \frac{2}{3} \right)}_{\frac{6}{3} + \frac{2}{3}} P_1 = \frac{1}{3} P_0 + 2 P_{2u} + \frac{4}{3} P_2$$

$$\left( \frac{1}{3} + 2 \right) P_{2u} = \frac{1}{3} P_{1u} + \frac{2}{3} P_2$$

$$\left( \frac{1}{3} + \frac{2}{3} \right) P_2 = \frac{1}{3} P_1 + 2 P_{3u} + \frac{4}{3} P_3$$

$$2 P_{3u} = \frac{1}{3} P_{2u} + \frac{2}{3} P_3$$

$$\left( \frac{4}{3} + \frac{2}{3} \right) P_3 = \frac{1}{3} P_2$$

## General Distribution Queues

### \* M/G/1 model

- Poisson input process
- No restriction on the probability distribution of the service times
- Service time mean =  $\frac{1}{\mu}$  with variance =  $\sigma^2$
- As long as  $\rho = \frac{\lambda}{\mu} < 1$ , we reach steady state

$$P_0 = 1 - \rho ; \boxed{L_q = \frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)}} ; L = \rho + L_q ; W_q = \frac{L_q}{\lambda}$$

→ Pollaczek Khintchine formula and  $W = W_q + \frac{1}{\mu}$

### \* M/D/1 model

- D: Degenerate: Constant Service Times (like coffee machine)
- let  $x$  be the constant service time. Then,
- $\mu = \frac{1}{x} ; \rho = \frac{\lambda}{\mu} ; \boxed{L_q = \frac{\rho^2}{2(1-\rho)}} : \text{Because there's no variation} \Rightarrow \sigma^2 = 0$
- Decreasing variability improves the system performance

### \* M/E<sub>k</sub>/1 Model

$$T \sim \text{Erlang}_k(\mu) \equiv T \sim \text{Gamma}(\alpha=k; \beta=\mu)$$

$$f(t) = \frac{(\lambda t)^k}{(k-1)!} \cdot t^{k-1} \cdot e^{-\lambda t}$$

$$E(T) = \frac{k}{\mu} \quad \text{Var}(T) = \frac{k}{\mu^2}$$

If  $T_1, T_2, \dots, T_k \stackrel{i.i.d}{\sim} \text{Exponential}(\lambda)$

Then  $T = T_1 + T_2 + \dots + T_k \sim E_k(\lambda)$

$\bullet k=1 \Rightarrow E_1(\lambda) \equiv \text{Exponential}(\lambda)$

$k \rightarrow \infty \Rightarrow \text{Var}(T) = 0 \Rightarrow \text{Degenerate Distribution}$

Erlang model is somewhere between M/D/1 and M/M/1 in terms of variability of service times.

We have;

$$L_q = \frac{\lambda^2 + \rho^2}{2(1-\rho)} = \frac{\rho^2 \left( \frac{1}{k} + 1 \right)}{2(1-\rho)} = \frac{\rho^2}{2(1-\rho)} \cdot \left( \frac{k+1}{k} \right)$$

17.7-4. Marsha operates an expresso stand. Customers arrive according to a Poisson process at a mean rate of 30 per hour. The time needed by Marsha to serve a customer has an exponential distribution with a mean of 75 seconds.

- (a) Use the M/G/1 model to find  $L_q$ ,  $W$ , and  $W_q$ .
  - (b) Suppose Marsha is replaced by an expresso vending machine that requires exactly 75 seconds for each customer to operate. Find  $L$ ,  $L_q$ ,  $W$ , and  $W_q$ .
  - (c) What is the ratio of  $L_q$  in part (b) to  $L_q$  in part (a)?
- T (d) Use trial and error with the Excel template for the M/G/1 model to see approximately how much Marsha would need to reduce her expected service time to achieve the same  $L_q$  as with the expresso vending machine.

17.7-5. Antonio runs a shoe repair store by himself. Customers arrive to bring a pair of shoes to be repaired according to a Poisson process at a mean rate of 1 per hour. The time Antonio requires to repair each individual shoe has an exponential distribution with a mean of 15 minutes.

- (a) Consider the formulation of this queueing system where the individual shoes (not pairs of shoes) are considered to be the customers. For this formulation, construct the rate diagram and develop the balance equations, but do not solve further.
  - (b) Now consider the formulation of this queueing system where the pairs of shoes are considered to be the customers. Identify the specific queueing model that fits this formulation.
  - (c) Calculate the expected number of pairs of shoes in the shop.
  - (d) Calculate the expected amount of time from when a customer drops off a pair of shoes until they are repaired and ready to be picked up.
- T (e) Use the corresponding Excel template to check your answers in parts (c) and (d).

17.7-4)  $\lambda = 30 \text{ customers/hour}$

$$\mu = \frac{60}{75} \cdot 60 = 48 \text{ customers/hour}$$

$T \sim \text{Exponential}(\mu)$

$$\text{Var}(T) = \frac{1}{\mu^2} = \frac{1}{48^2} = \sigma^2$$

$$a) \rho = \frac{\lambda}{\mu} = \frac{30}{48} = 0,625$$

$$P_0 = 1 - \rho = 1 - 0,625 = 0,375$$

$$L_q = \frac{30^2 \cdot \frac{1}{48^2} + 0,625^2}{2(1 - 0,625)} = 1,0417$$

$$W_q = \frac{1,0417}{30} = 0,0347 \text{ hours} = 2,08 \text{ min.}$$

$$L = 0,625 + 0,0417 = 1,6667$$

$$W = 2,08 + \frac{1}{48} \cdot 60 = 3,33 \text{ min.}$$

b) M/D/1 model ;  $\sigma^2 = 0$  ;  $P_0 = 0,375$

$$L_q = \frac{0,625^2}{2 \cdot (1 - 0,625)} = 0,5208 ; L = 0,625 + 0,5208 = 1,1458$$

$$W_q = \frac{0,5208}{30} \cdot 60 = 1,04 \text{ min.} ; W = 1,04 + \frac{60}{48} = 2,29 \text{ min.}$$

c)  $L_q$ -ratio =  $\frac{0,5208}{1,0417} = 0,5$  *→ This ratio is constant for any  $\mu$  M/D/1 : M/M/1*

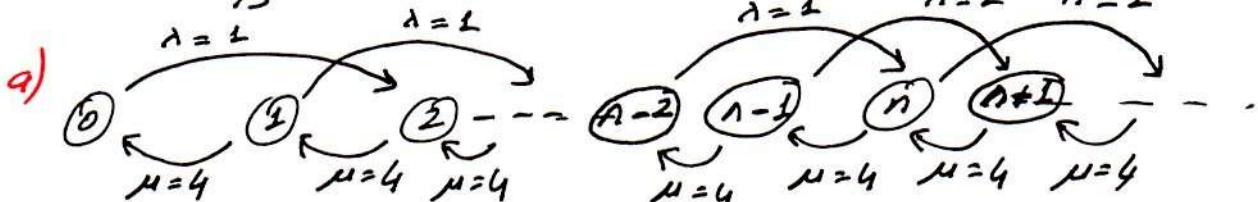
$$d) \frac{\frac{30^2}{\mu^2} + \frac{30^2}{\mu^2}}{2 \cdot (1 - \frac{30}{\mu})} = 0,5208$$

$$\frac{30^2}{\mu^2} = 0,5208 - \frac{15,625}{\mu}$$

$$\begin{aligned} 1728,11 &= \mu^2 - 30\mu \\ \mu^2 - 30\mu - 1728,11 &= 0 \\ \mu = 59,194 &\Rightarrow E(T) = \frac{1}{59,194} \text{ hours} \\ &= 61 \text{ seconds.} \end{aligned}$$

17.7-5)  $\lambda = 1$  pair of shoes / hour

$$\mu = \frac{60}{15} = 4 \text{ individual shoe / hour.}$$



$$P_0 = 4P_1$$

$$5P_1 = 4P_2$$

$$5P_2 = P_0 + 4P_3$$

!

$$5P_n = P_{n-2} + 4P_{n+1}$$

$$\sum_{n=0}^{\infty} P_n = 1$$

b)  $T_1, T_2 \sim \text{Exponential}(2, 2)$   $k=2 \text{ shoes/hour}$   $\rightarrow \text{New Rate for } T = T_1 + T_2$   
 $= 4 \text{ individual shoes/hour}$

$$T = T_1 + T_2 \sim E_2(2)$$

*= A pair of shoes / hour*

$M/E_2/1$  model

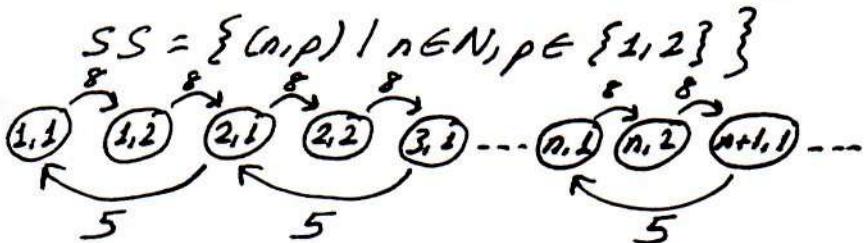
$$c) \angle = ? \quad L_9 = \frac{\frac{1^2}{2} + \left(\frac{1}{2}\right)^2}{2 \cdot \left(1 - \frac{1}{2}\right)} = 0,375; \quad \angle = 0,5 + 0,375 = 0,875$$

$$d) W = ? \quad W_0 = \frac{0,375}{1} = 0,375; \quad W = 0,375 + \frac{1}{4} = 0,625 \text{ hours} \\ = 37,5 \text{ min.}$$

**17.7-10.** Consider the  $E_2/M/1$  model with  $\lambda = 4$  and  $\mu = 5$ . This model can be formulated as a continuous time Markov chain by dividing each interarrival time into two consecutive phases, each having an exponential distribution with a mean of  $1/(2\lambda) = 0.125$ , and then defining the state of the system as  $(n, p)$ , where  $n$  is the number of customers in the system ( $n = 0, 1, 2, \dots$ ) and  $p$  indicates the phase of the next arrival (not yet in the system) ( $p = 1, 2$ ).

Construct the corresponding rate diagram (but do not solve further).

17. 7-10)  $\{x(t) : \text{states } (n, p)$   
of the system at time  $t \geq 0\}$



# Queering Networks

\* **Equivalence Property:** Assume an M/M/1 model with input parameter  $\lambda$  and  $\mu > \lambda$ . The steady state output of this service facility is also a Poisson process with parameter  $\lambda$ . Moreover, output Poisson Process is independent of the input Poisson Process.



!Equivalence Property does NOT hold for finite queues

\* Consider  $m$  queuing systems of a network. Since the Poisson processes are independent, the joint distribution of customers is;

$$P\{N_1 = n_1, N_2 = n_2, \dots, N_m = n_m\} = P_{n_1} \cdot P_{n_2} \cdots P_{n_m}$$

### Jackson Network

A jackson network is a system of  $m$  stations where station  $i$  has;

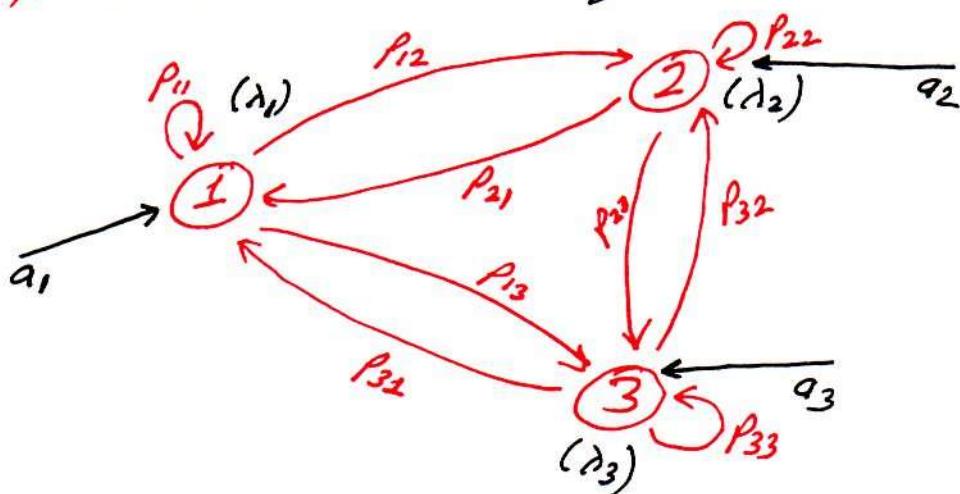
- (i) Infinite Queue
- (ii) Customers arriving outside  $\sim$  Poisson( $\lambda_i$ )  $i=1,2,\dots,m$
- (iii)  $s_i$  servers with identical  $\sim$  Exponential( $\mu_i$ )

We have;

$p_{ij}$  : Probability of a customer leaving facility  $i$  is rotated to facility  $j \in \{1, 2, \dots, m\}$

$q_i = 1 - \sum_{j=1}^m p_{ij}$  : Probability to quit from the system.

\* Consider the following network with 3 stations;



$a_1, a_2, a_3$  : Rate of customers coming from outside

$\lambda_1, \lambda_2, \lambda_3$  : Rate of TOTAL customers

$P_{ij}$  : Probability of customers rolling inside the network

$q_i$  : Probability of customers to quit from node  $i$

We have;

$$\lambda_j = a_j + \sum_{i=1}^m \lambda_i \cdot P_{ij} \quad j = 1, 2, \dots, m$$

Then;  $\lambda_1 = a_1 + p_{11} \lambda_1 + p_{21} \lambda_2 + p_{31} \lambda_3$

$$\lambda_2 = a_2 + p_{12} \lambda_1 + p_{22} \lambda_2 + p_{32} \lambda_3$$

$$\lambda_3 = a_3 + p_{13} \lambda_1 + p_{23} \lambda_2 + p_{33} \lambda_3$$

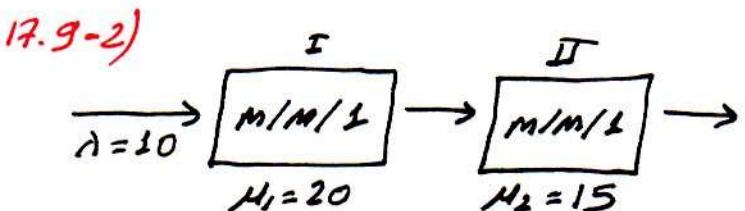
Or, equivalently;

$$\lambda = a + P^T \lambda \quad \text{where } \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}; a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}; P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

17.9-2. Consider a system of two infinite queues in series, where each of the two service facilities has a single server. All service times are independent and have an exponential distribution, with a mean of 3 minutes at facility 1 and 4 minutes at facility 2. Facility 1 has a Poisson input process with a mean rate of 10 per hour.

- (a) Find the steady-state distribution of the number of customers at facility 1 and then at facility 2. Then show the product form solution for the joint distribution of the number at the respective facilities.

- (b) What is the probability that both servers are idle?  
 (c) Find the expected total number of customers in the system and the expected total waiting time (including service times) for a customer.



a) (I)  $\rho = \frac{\lambda}{\mu_1} = \frac{10}{20} = 0,5$

$$P(N_1 = 0) = P_0^I = 1 - \rho = 0,5$$

$$P_n^I = (1 - \rho) \cdot \rho^n = 0,5 \cdot 0,5^n$$

$$(II) \rho = \frac{\lambda}{\mu_2} = \frac{10}{15} = 0,667$$

$$P_0^{II} = 0,333; P_n^{II} = 0,333 \cdot 0,667^n$$

Then;

$$P(N_1 = n_1, N_2 = n_2) = 0,167 \cdot 0,5^{n_1} \cdot 0,667^{n_2}$$

$$(II) \rho = \frac{\lambda}{\mu_2} = \frac{10}{15} = 0,667$$

$$b) P(N_1=0, N_2=0) = 0,167 \cdot 0,5^0 \cdot 0,667^0 = 0,167$$

$$c) L_1 = \frac{\lambda}{\mu_1 - \lambda} = \frac{10}{20-10} = 1 ; L_2 = \frac{\lambda}{\mu_2 - \lambda} = \frac{10}{15-10} = 2$$

$$W_1 = \frac{L_1}{\lambda} = \frac{1}{10} \text{ hours} = 6 \text{ min}; W_2 = \frac{L_2}{\lambda} = \frac{2}{10} \text{ hours} = 12 \text{ min.}$$

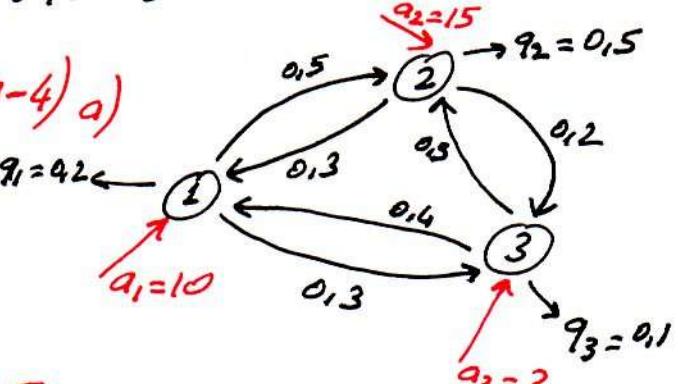
$$L = L_1 + L_2 = 1+2 = 3; W = W_1 + W_2 = 6+12 = 18 \text{ min.}$$

17.9-4. Consider a Jackson network with three service facilities having the parameter values shown below.

Facility $j$	$s_j$	$\mu_j$	$a_j$	$p_{ij}$		
				$i=1$	$i=2$	$i=3$
$j=1$	1	40	10	0	0.3	0.4
$j=2$	1	50	15	0.5	0	0.5
$j=3$	1	30	3	0.3	0.2	0

- T (a) Find the total arrival rate at each of the facilities.
- (b) Find the steady-state distribution of the number of customers at facility 1, facility 2, and facility 3. Then show the product form solution for the joint distribution of the number at the respective facilities.
- (c) What is the probability that all the facilities have empty queues (no customers waiting to begin service)?
- (d) Find the expected total number of customers in the system.
- (e) Find the expected total waiting time (including service times) for a customer.

17.9-4) a)



I

M/M/1

$$\mu_1 = 40$$

$$\lambda_1 = 30,12$$

II

M/M/1

$$\mu_2 = 50$$

$$\lambda_2 = 40,33$$

III

M/M/1

$$\mu_3 = 30$$

$$\lambda_3 = 20,04$$

$$\lambda_1 = 10 + 0,3\lambda_2 + 0,4\lambda_3$$

$$\lambda_2 = 15 + 0,5\lambda_1 + 0,5\lambda_3$$

$$\lambda_3 = 3 + 0,3\lambda_1 + 0,2\lambda_2$$

$$\begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \\ \left[ \begin{array}{ccc} 1 & -0,3 & -0,4 | 10 \\ -0,5 & 1 & -0,5 | 15 \\ -0,3 & -0,2 & 1 | 3 \end{array} \right] & \xrightarrow{0,5R_1+R_2} & \left[ \begin{array}{ccc} 1 & -0,3 & -0,4 | 10 \\ 0 & 0,85 & -0,7 | 20 \\ 0 & -0,29 & 0,88 | 6 \end{array} \right] \\ & \xrightarrow{0,3R_2+R_3} & \\ & \xrightarrow{0,29R_1+R_3} & \end{array}$$

$$\xrightarrow{\frac{0,29}{0,85} R_2+R_3} \left[ \begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & -0,3 & -0,4 | 10 \\ 0 & 0,85 & -0,7 | 20 \\ 0 & 0 & 0,64 | 12,82 \end{array} \right]$$

$$\text{Then; } 0,64\lambda_3 = 12,82$$

$$\underline{\underline{\lambda_3 = 20,04}}$$

$$0,85\lambda_2 - 0,7\lambda_3 = 20$$

$$0,85\lambda_2 = 20 + 0,7 \cdot 20,04$$

$$\underline{\underline{\lambda_2 = 40,33}}$$

$$\lambda_1 - 0,3\lambda_2 - 0,4\lambda_3 = 10$$

$$\lambda_1 = 10 + 0,3 \cdot 40,33 + 0,4 \cdot 20,04$$

$$\underline{\underline{\lambda_1 = 30,12}}$$

b)  $P_1 = \frac{30,12}{40} = 0,753 ; P_{n_1}^I = 0,247 \cdot 0,753^{n_1}$

$$P_2 = \frac{40,33}{50} = 0,807 ; P_{n_2}^{II} = 0,193 \cdot 0,807^{n_2}$$

$$P_3 = \frac{20,04}{30} = 0,680 ; P_{n_3}^{III} = 0,320 \cdot 0,680^{n_3}$$

$$P(N_1 = n_1, N_2 = n_2, N_3 = n_3) = 0,015 \cdot 0,753^{n_1} \cdot 0,807^{n_2} \cdot 0,680^{n_3}$$

c)  $P(N_1 \leq 1, N_2 \leq 1, N_3 \leq 1) = \sum_{n_3=0}^1 \sum_{n_2=0}^1 \sum_{n_1=0}^1 P(N_1 = n_1, N_2 = n_2, N_3 = n_3)$

$$= 0,015 \cdot 0,753^0 \cdot 0,807^0 \cdot 0,680^0 + 0,015 \cdot 0,753^1 \cdot 0,807^0 \cdot 0,680^0 + \dots + 0,015 \cdot 0,753^0 \cdot 0,807^1 \cdot 0,680^0$$

$$= 0,0798$$

d)  $L_1 = \frac{\lambda_1}{\mu_1 - \lambda_1} = \frac{30,12}{40 - 30,12} = 3,05$

$$L_2 = \frac{40,33}{50 - 40,33} = 4,17 ; L_3 = \frac{20,04}{30 - 20,04} = 2,02$$

$$L = L_1 + L_2 + L_3 = 9,23$$

e)  $W = \frac{L}{a_1 + a_2 + a_3} = \frac{9,23}{10 + 15 + 3} = 0,33 \text{ hours} \approx 20 \text{ min.}$

(Note that  $W \neq W_1 + W_2 + W_3$ )