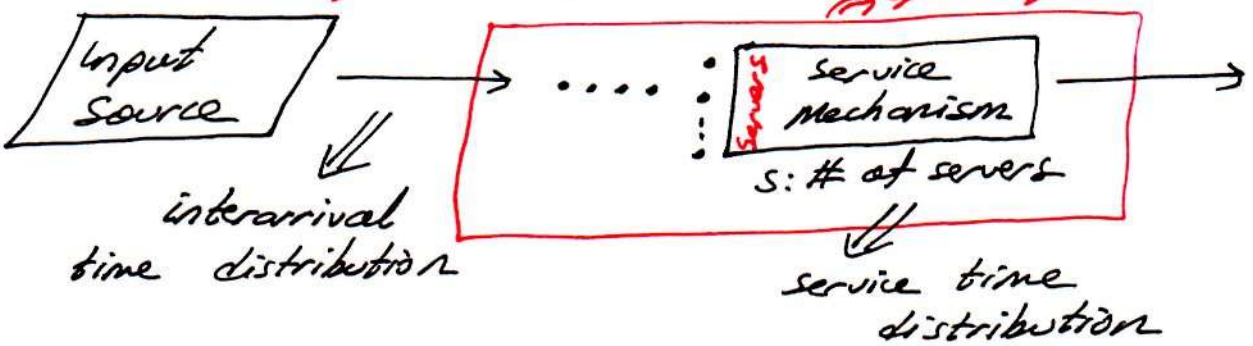
STOCHASTIC MODELS LECTURE NOTES	CHAPTERS 17.1-17.2-17.3-17.4-17.5

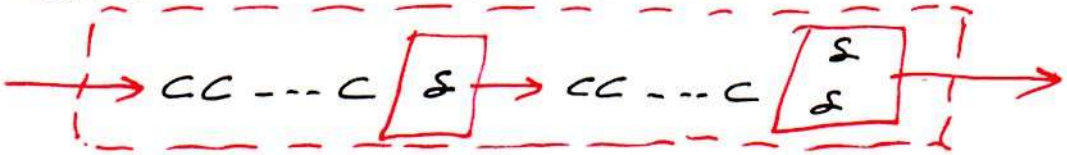
Queueing Theory - Basic Concepts



* POISSON PROCESS: Exponential (Markovian) interarrivals
* Unless stated otherwise, we'll assume ∞ input source.
* BALKING: Customer's decreasing probability to ENTER the system with increasing number of people in the system.
* RENEGING: Customer quit from the system due to queue.

Queue: Customers wait in a "single line" (independent of number of servers) to take service

* QUEUEING DISCIPLINE
  - FCFS: First come First Served (Our concern is this one)
  - SIRO: Service in Random Order (Ex: If there's Not queue)
  - LCFS: Last Come First Served (Ex: Same inventory uses this)
  - Priority Scheme (Ex: May be used in Army)
* SERIAL SERVICE MECHANISM:



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* SERVICE TIME: The time between "Starting of the service" for a customer and "End of the Service"

* SERVICE TIME DISTRIBUTION

M: Exponential (Markovian)

D: Degenerate (Constant Service times i.e. Machines)

E_k: Erlang with k: Shape Parameter

G: General (Any) distribution with known Mean and Variance.

Kendall-Lee Notation

Assume ∞ unless stated otherwise.

$$\_\_\_ / \_\_\_ / \_\_\_ \left(/ \_\_\_ / \_\_\_ \right)$$

Interarrival time Distribution

Service Time Distribution

s: # of servers

K: System Capacity (= s + Queueing Capacity)

N: Max # of Calling Population.

Example:

* M/M/3 : 3 identical servers with exponential interarrival and service times

* M/G/1/∞/10 : Single server, no restriction in service time distribution; Poisson process interarrivals; No restriction in system capacity; Calling population has 10 customers (potential)

Terminology and Notation:

- State of system: # of customers in the system
- Queue length = # of customers waiting for service

$$= \begin{pmatrix}\text{state of}\\ \text{system}\end{pmatrix} - \begin{pmatrix}\text{\# of customers}\\ \text{having service}\end{pmatrix}$$

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- $N(t)$: Number of customers in the system at time t. Note that, this is a CTMC.
- $P_n(t) = P\{N(t)=n\}$: The probability that there are n customers in the system at time t.
- s : # of servers
- λ_n : Mean arrival rate / unit time given that there are n customers in the system

(Note that unless stated otherwise, we'll assume $\lambda_n = \lambda$. λ_n's vary if there is Balking, or λ_n may be a function of n)

Note that, $RATE = \frac{1}{MEAN}$. For example, "Poisson process with 3 customers per hour" $\equiv$ "--- expected interarrival time is $\frac{1}{3}$ hours or 20 minutes"

- μ_n : Mean service rate / unit time given that there are n customers in the system

(In general, we'll assume $\mu_n = \mu$. Note that, in case of renaging, μ_n will vary because it does not matter for the system how a customer quits.)

T: Service time $\Rightarrow E(T) = \frac{1}{\mu}$ OR $\mu = \frac{1}{E(T)}$

- ρ : Utilization Factor ; $\rho = \frac{\lambda}{s \cdot \mu}$

We'll consider the case $\rho < 1$ because otherwise the queue will go to ∞. The closer ρ to 0, the fluent is the system. For single server; $\rho = 1 - P_0$

Steady State Condition

We'll calculate some performance measures for the system "in the long-run" OR "at steady state"

- $L = E[N(t)] = \sum_{n=0}^{\infty} n \cdot P_n$: Average (or Expected) number of customers in the system.
  (or $E(N)$)
- $L_q = E[N_q(t)] = \sum_{n=s}^{\infty} (n-s) \cdot P_n$: Expected queue length.
  (or $E(N_q)$)
- W : Waiting time of a customer in the system (queue + service)
- $\mathbb{W} = E(W)$: Average waiting time of a customer in the system.
- W_q : Waiting time of a customer in the queue
- $\mathbb{W}_q = E(W_q)$: Average waiting time of a customer in the queue

(Also $\mathbb{W}_s$: Average waiting time of a customer in the service)

Little's Formula (Result)

(i) $L = \lambda \cdot \mathbb{W}$ → let 3 customers/min. each waiting for 5 min. At steady state, $L = 3 \cdot 5 = 15$ customers will be in the system.

(ii) $L_q = \lambda \cdot \mathbb{W}_q$ (OR; $L_s = \lambda \cdot \mathbb{W}_s$)

(iii) $\mathbb{W} = \mathbb{W}_q + \frac{1}{\mu}$ $\left(E(\text{Service Time}) = \mathbb{W}_s = \frac{1}{\mu}\right)$

* Then, $L, \mathbb{W}, L_q, \mathbb{W}_q$ can all found if we know one of them. These formulas are valid for ALL queueing systems.

* If λ is NOT constant, replace λ with $\bar{\lambda} = \sum_{n=0}^{\infty} \lambda_n \cdot P_n$

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17.2-2.* Newell and Jeff are the two barbers in a barber shop they own and operate. They provide two chairs for customers who are waiting to begin a haircut, so the number of customers in the shop varies between 0 and 4. For $n = 0, 1, 2, 3, 4$, the probability P_n that exactly n customers are in the shop is $P_0 = \frac{1}{16}$, $P_1 = \frac{4}{16}$, $P_2 = \frac{6}{16}$, $P_3 = \frac{4}{16}$, $P_4 = \frac{1}{16}$.

(a) Calculate L. How would you describe the meaning of L to Newell and Jeff?

(b) For each of the possible values of the number of customers in the queueing system, specify how many customers are in the queue. Then calculate L_q. How would you describe the meaning of L_q to Newell and Jeff?

(c) Determine the expected number of customers being served.

(d) Given that an average of 4 customers per hour arrive and stay to receive a haircut, determine W and W_q. Describe these two quantities in terms meaningful to Newell and Jeff.

(e) Given that Newell and Jeff are equally fast in giving haircuts, what is the average duration of a haircut?

17.2-2)

n	0	1	2	3	4
P_n	1/16	4/16	6/16	4/16	1/16

a) $L = E(N) = \sum_{n=0}^{4} n \cdot P_n$

$= 0 \cdot \frac{1}{16} + 1 \cdot \frac{4}{16} + \cdots + 4 \cdot \frac{1}{16} = 2$

b) $s = 2$; $L_q = E(N_q) = \sum_{n=2}^{4} (n-2) \cdot P_n$

$= 0 \cdot \frac{6}{16} + 1 \cdot \frac{4}{16} + 2 \cdot \frac{6}{16} = \frac{3}{8} = 0{,}375$

c) $L_s = E(N_s) = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot \left(\sum_{n=2}^{\infty} P_2 \right)$

$= 0 \cdot \frac{1}{16} + 1 \cdot \frac{4}{16} + 2 \cdot \left(\frac{6}{16} + \frac{4}{16} + \frac{1}{16} \right) = 1{,}625$

(OR ; $L_s = L - L_q = 2 - 0{,}375 = 1{,}625$)

d) $\lambda = 4$ $\quad W = \frac{L}{\lambda} = \frac{2}{4} = 0{,}5$ hours

$W_q = \frac{L_q}{\lambda} = \frac{0{,}375}{4} = 0{,}09375$ hours

e) $E(\text{Service time}) = \frac{1}{\mu} = W - W_q = 0{,}5 - 0{,}09375$

$= 0{,}40625$ hours

17.2-5. Midtown Bank always has two tellers on duty. Customers arrive to receive service from a teller at a mean rate of 40 per hour. A teller requires an average of 2 minutes to serve a customer. When both tellers are busy, an arriving customer joins a single line to wait for service. Experience has shown that customers wait in line an average of 1 minute before service begins.

(a) Describe why this is a queueing system.

(b) Determine the basic measures of performance—W_q, W, L_q, and L—for this queueing system. (*Hint*: We don't know the probability distributions of interarrival times and service times for this queueing system, so you will need to use the relationships between these measures of performance to help answer the question.)

17.2-5) $\lambda = 40$ / hour

$\mu = \frac{60}{2} = 30$ / hour ; $s = 2$ servers

$W_q = 1$ minutes.

b) $W = W_q + W_s = 1 + 2 = 3$ minutes

$L = \lambda \cdot W = 40 \cdot \frac{3}{60} = 2$ customers

$L_q = \lambda \cdot W_q = 40 \cdot \frac{1}{60} = 0{,}667$ customers

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17.2-8. Consider a single-server queueing system with *any* service-time distribution and *any* distribution of interarrival times (the *GI/G/1* model). Use only basic definitions and the relationships given in Sec. 17.2 to verify the following general relationships:
(a) $L = L_q + (1 - P_0)$.
(b) $L = L_q + \rho$.
(c) $P_0 = 1 - \rho$.

17.2-8) $s = 1$; $G/G/1$

a) $L_s = 0 \cdot P_0 + 1 \cdot \sum_{n=1}^{\infty} P_n$

$\sum_{n=1}^{\infty} P_n = \sum_{n=0}^{\infty} P_n - P_0 = 1 - P_0$

$L_s = 1 - P_0$

$\boxed{L = L_q + L_s = L_q + (1 - P_0)}$

b) $L = \lambda \cdot W$

$L_q = \lambda \cdot W_q$

$L_s = \lambda \cdot W_s = \lambda \cdot \frac{1}{\mu} = \frac{\lambda}{\mu}$

since $s = 1$; $\rho = \frac{\lambda}{\mu} = s$

Then $\boxed{L = L_q + L_s = L_q + \rho}$

c) $L_s = 1 - P_0$

$\boxed{P_0 = 1 - L_s = 1 - \rho}$

Exponential Distribution

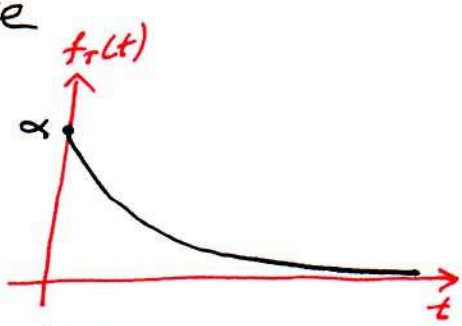
$T \sim \text{Exponential}(\alpha)$

$$f_T(t) = \begin{cases} \alpha \cdot e^{-\alpha t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

α: Rate

$P(T \leq t) = F(t) = 1 - e^{-\alpha t}$

$E(T) = \frac{1}{\alpha}$ $\quad Var(T) = \frac{1}{\alpha^2}$



Property 1: Strictly decreasing function

* Property 2 : LACK of MEMORY (Memoriless)

$P\{T > t+s \mid T > s\} = P\{T > t\}$

Ex: T: interarrival time of the customers

let 5 customers/hour come to a coffee on the average.

If there were no customers for two hours, WPT there'll be no customers next half hour?

$P(T > 2{,}5 \mid T > 2) = P(T > 0{,}5) = e^{-5 \cdot 0{,}5} = 0{,}0821$

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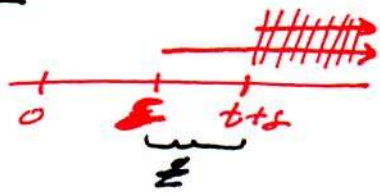
Proof of Memoriless Remember; $P(A|B) = \dfrac{P(AB)}{P(B)}$

Then; $P(T > t+s \mid T > s) = \dfrac{P(T > t+s,\ T > s)}{P(T > s)}$

$= \dfrac{P(T > t+s)}{P(T > s)} = \dfrac{e^{-\alpha(t+s)}}{e^{-\alpha s}} = e^{-\alpha t} = P(T > t)$



Property 3: Minimum of Exponentials

Minimum of several indipendent exponential random variables has an exponential distribution. Namely,

let $T_i \sim \text{Exponential}(\alpha_i)$ $i \in \{1, 2, \dots, n\}$

$U = \min(T_1, T_2, \dots, T_n)$

Then; $P(U > t) = P(T_1 > t, T_2 > t, \dots, T_n > t)$ by independence,

$= P(T_1 > t) \cdot P(T_2 > t) \cdot \dots \cdot P(T_n > t) = e^{-\alpha_1 t} \cdot e^{-\alpha_2 t} \cdot \dots \cdot e^{-\alpha_n t}$

$= e^{-\sum_{i=1}^{n} \alpha_i t}$ so; $U \sim \text{Exponential}(\sum \alpha_i)$

ALSO; $P(U = T_j) = \dfrac{\alpha_j}{\sum_{i=1}^{n} \alpha_i}$

Ex There are 4 horses in a horserace. They finish the tour on the average 20, 15, 12 and 10 minutes respectively with exponential runtimes.

a) WPT the race finishes in 5 minutes?

b) WPT the third horse win the race?

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Ans; Rates are: $\frac{60}{20} = 3 = \alpha_1$; $\frac{60}{15} = 4 = \alpha_2$; $\frac{60}{12} = 5 = \alpha_3$; $\frac{60}{10} = 6 = \alpha_4$

Respectively. We have; $T_i \sim \text{Exponential}(\alpha_i)$

U = Finishing time of the race

$U = \text{Min}(T_1, T_2, T_3, T_4)$

$U \sim \text{Exponential}(3+4+5+6 = 18)$

a) $P(U < 5\text{min.}) = P(U < \frac{5}{60} \text{ hours}) = F(\frac{5}{60}) = 1 - e^{-18 \cdot \frac{5}{60}}$

$= 1 - e^{-1,5} = 0,777$

b) $P(U = T_3) = \frac{\alpha_3}{\Sigma \alpha_i} = \frac{5}{18} = 0,278$

* Property 4: Relation With Poisson Process

Let $N(t)$: # of events until time t

T: Time between events (or equivalently, time until next event)

We have; Remember, $P(N(t)=n) = \frac{e^{-(\lambda t)} \cdot (\lambda t)^n}{n!}$

$N(t) \sim \text{Poisson}(\lambda t) \Leftrightarrow T \sim \text{Exponential}(\lambda)$

Because; $P(N(t)=0) = \frac{(\lambda t)^0 \cdot e^{-\lambda t}}{0!} = e^{-\lambda t} = P(T > t)$

Exe Remember the cafe example of Memoriless

$T \sim \text{Exponential}(\alpha = 5 \text{ customers/hour})$

a) WPT there will be 12 customers within next 2,5 hours?

b) WPT next customer will be at least 10 minutes later?

Ans: $N(t) = \text{Poisson}(\lambda t)$; $\lambda = 5$

a) $P(N(2{,}5) = 12) = \dfrac{e^{-5.2{,}5} \cdot (5.2{,}5)^{12}}{12!} = 0{,}1132$

b) $P\left(N\left(\frac{10}{60}\right) = 0\right) = e^{-\left(5 \cdot \frac{10}{60}\right)} = 0{,}4346$

Property 5: Constant Limiting Probability

$$\lim_{\Delta t \to 0} \frac{P(T \le t + \Delta t \mid T > t)}{\Delta t} = \alpha$$

Note that this is just a corollary of memorilessness.

* Property 6: Unaffected by Aggregation or Disaggregation

Aggregation

Independent Poisson Processes

$\text{Poisson}(\lambda_1)$
$\text{Poisson}(\lambda_2)$
$\vdots$
$\text{Poisson}(\lambda_n)$

$\xrightarrow{\oplus} \text{Poisson}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$

* Note that this is by minimum of exponentials. "Which customer will come first?"

Disaggregation: Converse is also true

$$\text{Poisson}(\lambda) \xrightarrow{\oplus} \begin{cases} \xrightarrow{p_1} \sim \text{Poisson}(p_1 \cdot \lambda) \\ \xrightarrow{p_2} \sim \text{Poisson}(p_2 \cdot \lambda) \\ \vdots \\ \xrightarrow{p_n} \sim \text{Poisson}(p_n \lambda) \end{cases} \quad \text{where } \sum_{i=1}^{n} p_i = 1$$

* Note that, this is by extension of min. of exponentials. Here, Just Let $p_j = \dfrac{\lambda_j}{\sum \lambda_j}$

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Ex: Yonca oil factory gets 10 orders of sunflower oil 2 orders of Corn oil and 3 orders of Canola oil on the average in a month. If orders follow independent Poisson Processes,

a) WPT there will be 21 orders Next Month?

b) WPT Next order will be Canola oil?

Ans; a) N: # of orders per month

$N \sim \text{Poisson}(10+2+3=15)$

$$P(N=21) = \frac{e^{-15} \cdot 15^{21}}{21!} = 0{,}0299$$

b) U: Time until Next Customer

$$P(U = T_3) = \frac{\lambda_3}{\Sigma \lambda_i} = \frac{3}{10+2+3} = \frac{3}{15} = 0{,}2$$

Ex: Your car is broken down 5 times a year on the average, by Poisson process. with 0,6 probability, it is about motor, 0,3 probability about wheel and remaining by other factors. WPT you'll have motor problem 4 times next year?

Ans: Let N: # of car's broken down.

N_1: # of Motor's broken down

$N \sim \text{Poisson}(5) \Rightarrow N_1 \sim \text{Poisson}(5 \cdot 0{,}6 = 3)$

Then, $P(N_1 = 4) = \dfrac{e^{-3} \cdot 3^4}{4!} = 0{,}168$

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17.4-1. Suppose that a queueing system has two servers, an exponential interarrival time distribution with a mean of 2 hours, and an exponential service-time distribution with a mean of 2 hours for each server. Furthermore, a customer has just arrived at 12:00 noon.

(a) What is the probability that the next arrival will come (i) before 1:00 P.M., (ii) between 1:00 and 2:00 P.M., and (iii) after 2:00 P.M.?

(b) Suppose that no additional customers arrive before 1:00 P.M. Now what is the probability that the next arrival will come between 1:00 and 2:00 P.M.?

(c) What is the probability that the number of arrivals between 1:00 and 2:00 P.M. will be (i) 0, (ii) 1, and (iii) 2 or more?

(d) Suppose that both servers are serving customers at 1:00 P.M. What is the probability that *neither* customer will have service completed (i) before 2:00 P.M., (ii) before 1:10 P.M., and (iii) before 1:01 P.M.?

17.4-1) let T: interarrival time

$T \sim \text{Exponential}\left(\lambda = \frac{1}{2}\right)$

S_i: Service time $i = 1, 2$

$S_i \sim \text{Exponential}\left(\mu = \frac{1}{2}\right)$

Arrival of first customer: 12:00

a) (i) $P(T < 1) = F(1) = 1 - e^{-\frac{1}{2}\cdot 1} = 0{,}3935$

(ii) $P(1 < T < 2) = F(2) - F(1) = (1 - e^{-\frac{1}{2}\cdot 2}) - (1 - e^{-\frac{1}{2}\cdot 1})$

$= e^{-0{,}5} - e^{-1} = 0{,}2387$

(iii) $P(T > 2) = e^{-\frac{1}{2}\cdot 2} = 0{,}3679$

b) $P(T < 2 \mid T > 1) = P(T < 1) = 0{,}3935$

↳ by Memoriless

c) $N(t) \sim \text{Poisson}\left(\frac{1}{2} t\right)$

(i) $P(N(1) = 0) = P(T > 1) = e^{-\frac{1}{2}\cdot 1} = 0{,}6065$

(ii) $P(N(1) = 1) = \frac{e^{-1/2}(1/2)^1}{1!} = 0{,}3032$

(iii) $P(N(1) \geq 2) = 1 - P(N(1) \leq 1) = 1 - [0{,}6065 + 0{,}3032] = 0{,}0902$

d) Let $U = \text{Min}(S_1, S_2)$

Then $U \sim \text{Exponential}\left(\frac{1}{2} + \frac{1}{2} = 1\right)$

If $U > t$ then Neither Customer is served until time t.

Then; (i) $P(U > 1) = e^{-1} = 0{,}3679$

(ii) $P\left(U > \frac{10}{60}\right) = e^{-\frac{10}{60}} = 0{,}8465$

(iii) $P\left(U > \frac{1}{60}\right) = e^{-\frac{1}{60}} = 0{,}9835$

17.4-5. A queueing system has three servers with expected service times of 20 minutes, 15 minutes, and 10 minutes. The service times have an exponential distribution. Each server has been busy with a current customer for 5 minutes. Determine the expected remaining time until the next service completion.

17.4-6. Consider a queueing system with two types of customers. Type 1 customers arrive according to a Poisson process with a mean rate of 5 per hour. Type 2 customers also arrive according to a Poisson process with a mean rate of 5 per hour. The system has two servers, both of which serve both types of customers. For both types, service times have an exponential distribution with a mean of 10 minutes. Service is provided on a first-come-first-served basis.

(a) What is the probability distribution (including its mean) of the time between consecutive arrivals of customers of any type?

(b) When a particular type 2 customer arrives, she finds two type 1 customers there in the process of being served but no other customers in the system. What is the probability distribution (including its mean) of this type 2 customer's waiting time in the queue?

17.4-7. Consider a two-server queueing system where all service times are independent and identically distributed according to an exponential distribution with a mean of 10 minutes. Service is provided on a first-come-first-served basis. When a particular customer arrives, he finds that both servers are busy and no one is waiting in the queue.

(a) What is the probability distribution (including its mean and standard deviation) of this customer's waiting time in the queue?

(b) Determine the expected value and standard deviation of this customer's waiting time in the system.

(c) Suppose that this customer still is waiting in the queue 5 minutes after its arrival. Given this information, how does this change the expected value and the standard deviation of this customer's total waiting time in the system from the answers obtained in part (*b*)?

17.4-5) $S_1 \sim \text{Exponential}(3)$

$S_2 \sim \text{Exponential}(4)$

$S_3 \sim \text{Exponential}(6)$

$U \sim \text{Exponential}(3+4+6=13)$

$T = U + \frac{5}{60}$ hours

$E(T) = E(U + \frac{5}{60}) = E(U) + \frac{5}{60}$

$= \frac{1}{13} + \frac{5}{60} = 0{,}16$ hours

$= 9{,}62$ minutes

17.4.6) $T_i \sim \text{Exponential}(5)$ $i = 1, 2$ (Type)

$S_i = \text{Exponential}(6)$ $i = 1, 2$ (Server)

a) $U = \min(T_1, T_2)$

$U \sim \text{Exponential}(5+5=10)$

$E(U) = \frac{1}{10} = 6$ minutes

b) $W_q = \min(S_1, S_2)$

$W_q \sim \text{Exponential}(6+6=12)$

$E(W_q) = \frac{1}{12}$ hours $= 5$ minutes

17.4-7) $S_i \sim \text{Exponential}(6)$ $i = 1, 2$ (Server)

a) $U \sim \text{Exponential}(6+6=12)$

$E(U) = \frac{1}{12}$ hours ; $Var(U) = \frac{1}{12^2}$; $Std\text{-}Dev(U) = \frac{1}{12}$ hours

$= 5$ minutes $\qquad = 5$ minutes

b) $W = U + S_i$ (servers are identical, No matter if $i = 1$ or 2)

$E(W) = E(U + S_i) = E(U) + E(S_i) = \frac{1}{12} + \frac{1}{6} = \frac{3}{12}$ hours $= 15$ minutes

$Var(W) = Var(U + S_i) = Var(U) + Var(S_i) = \frac{1}{12^2} + \frac{1}{6^2} = \frac{5}{144} \Rightarrow Std\text{-}dev(U)$

$= \sqrt{\frac{5}{144}} = 0{,}186$ hours

$= 11{,}18$ min.

c) $T = 5 + W$ (minutes)

$E(T) = E(5 + W) = 5 + 15 = 20$ min ; $Std\text{-}Dev(T) = 11{,}18$ min

(Variance do NOT change)

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Queueing System: Customers & Servers

To give the main idea, the questions to be asked are: "what is the service?" and "who are being served and performing a queue to get service?"

Basic service systems are as follows:

* Commercial Service Systems: Outside customers recieve service from commercial organizations.

Servers: Barkers, bank tellers, checkout stands ... etc.

* Transportation Service Systems:

Vehicles are customers: tollbooth, traffic light, truck or ship waiting to be loaded, airplanes waiting to land or take off, cars searching for a parking lot

Vehicles are servers: taxicabs (but it depends. who is forming the queue? Taxi, or the customers?) Fire trucks, Elevators ... etc.

* Internal service Systems:

Materials-handling units (the servers) move loads (the customers)
Machine maintenance system (the servers) repair machines (the customers)
Quality Control inspectors (the servers) inspection items (the customers)
Machines (servers) jobs being processed (the customers)

* Social Service Systems;

The judges (servers) the cases waiting (customers)
Legislative system (servers) bills waiting (customers)
Hospital Emergency Rooms, Ambulances, X-ray Machines (servers) patients (customers)

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Birth and Death Processes

Birth: Arrival of a new Customer

Death: Departure of a customer from the system

$N(t)$: State of the system at time t

* The idea is to consider the world as serving to the living organisms. A process $N(t)$ is called a "Birth and Death" process if;

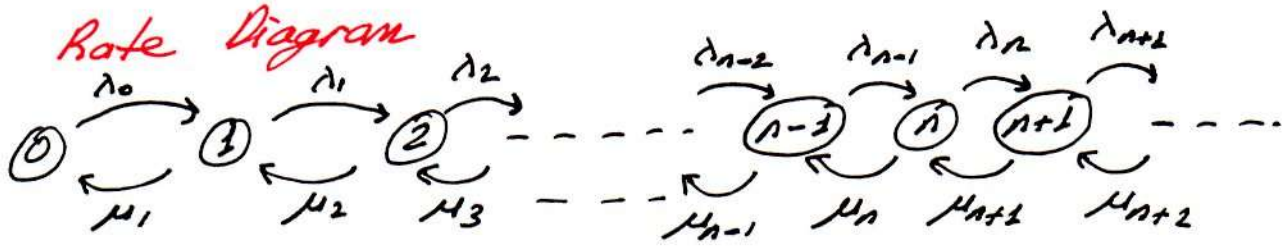
(i) Births $\sim$ Exponential (λ_n) → Interarrival Time distribution

(ii) Deaths $\sim$ Exponential (μ_n) → Service Time distribution

(iii) Births and Deaths are independent

* Birth and Death process is a CTMC. Given state n; the next event (or state) is either $n-1$ or $n+1$ because two events CANNOT happen at the same time.

Rate Diagram



Balance Equations

I'll use the Notation P_n corresponding to steady state probabilities ~~equations~~ π_n in discrete case.

$\boxed{\text{Rate OUT} = \text{Rate IN}}$ principle

State	Equation
0	$\lambda_0 P_0 = \mu_1 P_1$
1	$(\lambda_1+\mu_1) P_1 = \lambda_0 P_0 + \mu_2 P_2$
2	$(\lambda_2+\mu_2) P_2 = \lambda_1 P_1 + \mu_3 P_3$
⋮	⋮
n	$(\lambda_n+\mu_n) P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}$
⋮	⋮

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By back substitution, we have;

$$P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$P_2 = \frac{\lambda_0 \cdot \lambda_1}{\mu_1 \cdot \mu_2} P_0$$

$$\vdots$$

$$P_n = \frac{\lambda_0 \cdot \lambda_1 \cdot \ldots \cdot \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n} P_0 = \frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} P_0$$

Right hand Rule!

$$\vdots$$

and $P_0 + P_1 + P_2 + \ldots + P_n + \ldots = \sum_{i=0}^{\infty} P_i = 1$ Total Prob. is 1.

* To simplify the steady state equations,

Define $C_0 = 1$ and $C_n = \dfrac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i}$ and so,

$$\boxed{P_n = C_n \cdot P_0} \quad n = 0, 1, \ldots$$

Sometimes, some series formula is necessary here.

From the last equation;

$$\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} C_n \cdot P_0 = P_0 \cdot \sum_{n=0}^{\infty} C_n = 1 \Rightarrow \boxed{P_0 = \left[\sum_{n=0}^{\infty} C_n\right]^{-1}}$$

* Also, Remember some formula and Little's Result;

$$L = \sum_{n=0}^{\infty} n \cdot P_n \; ; \quad L_q = \sum_{n=s}^{\infty} (n-s) \cdot P_n \; ; \; L_s = \sum_{n=0}^{s-1} n \cdot P_n + s \cdot \sum_{n=s}^{\infty} P_n$$

$$L = \bar{\lambda} \cdot W \; ; \; L_q = \bar{\lambda} \cdot W_q \; ; \; L_s = \bar{\lambda} \cdot W_s \; ; \; \bar{\lambda} = \sum_{n=0}^{\infty} \lambda_n \cdot P_n$$

$$L = L_q + L_s \; ; \; W = W_q + W_s \; ; \; W_s = \frac{1}{\mu} \; ; \; \rho = \frac{\lambda}{s \cdot \mu}$$

* To reach steady state, necessary and sufficient condition is;

$\rho < 1$ OR equivalently, $\sum_{n=0}^{\infty} C_n < \infty$

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17.5-3. Consider the birth-and-death process with the following mean rates. The birth rates are $\lambda_0 = 2, \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$, and $\lambda_n = 0$ for $n > 3$. The death rates are $\mu_1 = 3, \mu_2 = 4, \mu_3 = 1$, and $\mu_n = 2$ for $n \geq 4$.

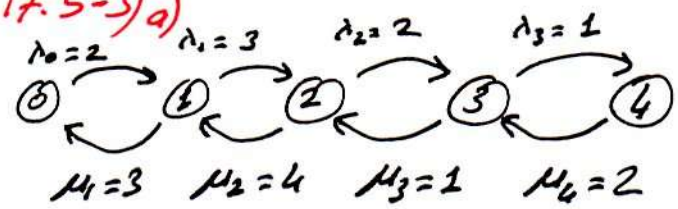
(a) Construct the rate diagram for this birth-and-death process.

(b) Develop the balance equations.

(c) Solve these equations to find the steady-state probability distribution $P_0, P_1, \ldots$.

(d) Use the general formulas for the birth-and-death process to calculate $P_0, P_1, \ldots$. Also calculate L, L_q, W, and W_q.

17.5-3) a)



b)
$$2P_0 = 3P_1$$
$$6P_1 = 2P_0 + 4P_2$$
$$6P_2 = 3P_1 + P_3$$
$$2P_3 = 2P_2 + 2P_4$$
$$2P_4 = P_3$$
$$P_0 + P_1 + P_2 + P_3 + P_4 = 1$$

c)
$$P_1 = \frac{2}{3} P_0 = 0{,}182$$
$$P_2 = \frac{2.3}{4.3} P_0 = \frac{1}{2} P_0 = 0{,}136$$
$$P_3 = \frac{2.3.2}{3.4.1} P_0 = P_0 = 0{,}273$$
$$P_4 = \frac{2.3.2.1}{3.4.1.2} P_0 = \frac{1}{2} P_0 = 0{,}136$$
$$P_0 = \left[1 + \frac{2}{3} + \frac{1}{2} + 1 + \frac{1}{2}\right]^{-1} = 0{,}273$$

n	0	1	2	3	4
P_n	0,182	0,136	0,273	0,136	0,273

d)
$$L = \sum_{n=0}^{4} n.P_n = 0.\,0{,}182 + 1.\,0{,}136 + \cdots + 4.\,0{,}273 = 2{,}182$$
$$\bar{\lambda} = \sum \lambda_n . P_n = 2.\,0{,}182 + 3.\,0{,}136 + 2.\,0{,}273 + 1.\,0{,}136 = 1{,}454$$
$$W = \frac{L}{\bar{\lambda}} = \frac{2{,}182}{1{,}454} = 1{,}501$$

Assuming single server;
$$L_q = \sum_{n=1}^{4} (n-1).P_n = 0.\,0{,}136 + 1.\,0{,}273 + 2.\,0{,}136 + 3.\,0{,}273 = 1{,}364$$
$$W_q = \frac{L_q}{\bar{\lambda}} = \frac{1{,}364}{1{,}454} = 0{,}938$$

Also, we have; $L_s = L - L_q = 2{,}182 - 1{,}364 = 0{,}818$
$$W_s = \frac{L_s}{\bar{\lambda}} = \frac{0{,}818}{1{,}454} = 0{,}563$$

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17.5-5.* A service station has one gasoline pump. Cars wanting gasoline arrive according to a Poisson process at a mean [redacted] of 15 minutes [redacted]. However, if the pump already is being used, these potential customers may *balk* (drive on to another service station). In particular, if there are n cars already at the service station, the probability that an arriving potential customer will balk is $n/3$ for $n = 1, 2, 3$. The time required to service a car has an exponential distribution with a mean of 4 minutes.

(a) Construct the rate diagram for this queueing system.
(b) Develop the balance equations.
(c) Solve these equations to find the steady-state probability distribution of the number of cars at the station. Verify that this solution is the same as that given by the general solution for the birth-and-death process.
(d) Find the expected waiting time (including service) for those cars that stay.

17.5-6. A maintenance person has the job of keeping two machines in working order. The amount of time that a machine works before breaking down has an exponential distribution with a mean of 10 hours. The time then spent by the maintenance person to repair the machine has an exponential distribution with a mean of 8 hours.

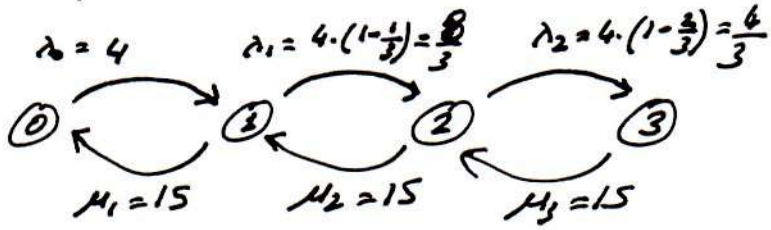
(a) Show that this process fits the birth-and-death process by defining the states, specifying the values of the λ_n and μ_n, and then constructing the rate diagram.
(b) Calculate the P_n.
(c) Calculate L, L_q, W, and W_q.
(d) Determine the proportion of time that the maintenance person is busy.
(e) Determine the proportion of time that any given machine is working.
(f) Refer to the nearly identical example of a *continuous time Markov chain* given at the end of Sec. 16.8. Describe the relationship between continuous time Markov chains and the birth-and-death process that enables both to be applied to this same problem.

17.5-5) $\lambda = 4$ cars / hour

$P(\text{Balk}) = \frac{n}{3}$ for $n = 1, 2, 3$

$\mu = 15$ cars / hour

a)



b-c) $P_1 = \frac{4}{15} \cdot P_0 = 0{,}2023$

$$P_2 = \frac{4 \cdot \frac{8}{3} P_0}{15 \cdot 15} = \frac{32}{675} P_0 = 0{,}0360$$

$$P_3 = \frac{4 \cdot \frac{8}{3} \cdot \frac{4}{3}}{15 \cdot 15 \cdot 15} P_0 = \frac{128}{30375} P_0 = 0{,}0031$$

Then; $P_0 = \left[1 + \frac{4}{15} + \frac{32}{675} + \frac{128}{30375}\right]^{-1} = 0{,}7586$

n	0	1	2	3
P_n	0,7586	0,2023	0,0360	0,0031

d) $W = ?$

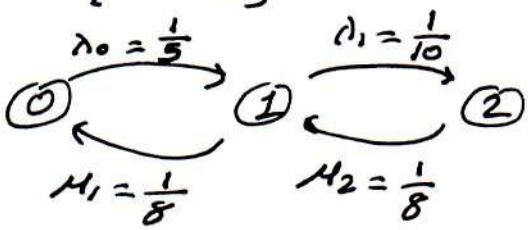
$$\bar{\lambda} = \sum \lambda_n P_n = 4 \cdot 0{,}7586 + \frac{8}{3} \cdot 0{,}2023 + \frac{4}{3} \cdot 0{,}0360 = 3{,}622$$

$$L = \sum n \cdot P_n = 0 \cdot 0{,}7586 + 1 \cdot 0{,}2023 + 2 \cdot 0{,}0360 + 3 \cdot 0{,}0031 = 0{,}284$$

$$W = \frac{L}{\bar{\lambda}} = \frac{0{,}284}{3{,}622} = 0{,}078 \text{ hours} = 4{,}7 \text{ minutes}$$

17.5-6) a) $\{X_t$: # of machines down at time $t \geq 0\}$

$SS = \{0, 1, 2\}$



b) $P_1 = \dfrac{1/5}{1/8} P_0 = \dfrac{8}{5} P_0$

$P_2 = \dfrac{1/5 \cdot 1/10}{1/8 \cdot 1/8} = \dfrac{32}{25} P_0$

$\Rightarrow$ $P_0 = [1 + 8/5 + 32/25]^{-1} = 0{,}258$

$P_1 = 0{,}412$

$P_2 = 0{,}330$

c) $L = \sum n \cdot P_n = 0 \cdot 0{,}258 + 1 \cdot 0{,}412 + 2 \cdot 0{,}330 = 1{,}072$

$\bar{\lambda} = \sum n \cdot \lambda_n = 0{,}258 \cdot \frac{1}{5} + 0{,}412 \cdot \frac{1}{10} = 0{,}093$

$W = \dfrac{L}{\bar{\lambda}} = \dfrac{1{,}072}{0{,}093} = 11{,}53$ hours

$W_s = \dfrac{1}{\mu} = \dfrac{1}{1/8} = 8$ hours ; $W_q = W - W_s = 11{,}53 - 8 = 3{,}53$ hours

$L_q = \bar{\lambda} \cdot W_q = 0{,}093 \cdot 3{,}53 = 0{,}328$

d) $1 - P_0 = 1 - 0{,}258 = 0{,}742$

e) $P_2 + \frac{1}{2} P_1 = 0{,}33 + \frac{1}{2} \cdot 0{,}412 = 0{,}536$

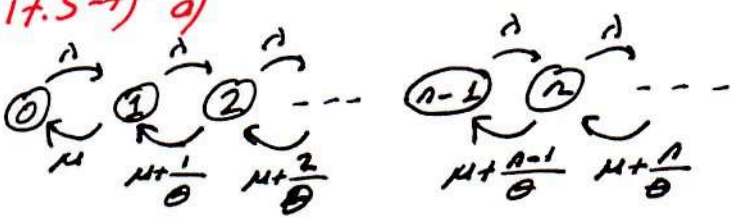
f) Birth and death process is already a CTMC.

17.5-7. Consider a single-server queueing system where interarrival times have an exponential distribution with parameter λ and service times have an exponential distribution with parameter μ. In addition, customers *renege* (leave the queueing system without being served) if their waiting time in the queue grows too large. In particular, assume that the time each customer is willing to wait in the queue before reneging has an exponential distribution with a mean of $1/\theta$.

(a) Construct the rate diagram for this queueing system.

(b) Develop the balance equations.

17.5-7) a)



b) $P_1 = \dfrac{\lambda}{\mu} P_0$

$P_2 = \dfrac{\lambda^2}{\mu \cdot (\mu + \frac{1}{\theta})} P_0$

$\vdots$

$P_n = \dfrac{\lambda^n}{\prod_{i=0}^{n-1} (\mu + \frac{i}{\theta})} P_0$

$\vdots$

$\sum_{n=0}^{\infty} P_n = 1$

17.5-9. A department has one word-processing operator. Documents produced in the department are delivered for word processing according to a Poisson process with an expected interarrival time of 20 minutes. When the operator has just one document to process, the expected processing time is 15 minutes. When she has more than one document, then editing assistance that is available reduces the expected processing time for each document to 10 minutes. In both cases, the processing times have an exponential distribution.

(a) Construct the rate diagram for this queueing system.

(b) Find the steady-state distribution of the number of documents that the operator has received but not yet completed.

(c) Derive L for this system. (*Hint:* Refer to the derivation of L for the $M/M/1$ model at the beginning of Sec. 17.6.) Use this information to determine L_q, W, and W_q.

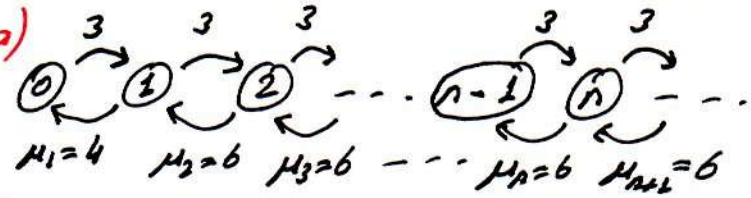
17.5-10. Customers arrive at a queueing system according to a Poisson process with a mean arrival rate of 2 customers per minute. The service time has an exponential distribution with a mean of 1 minute. An unlimited number of servers are available as needed so customers never wait for service to begin. Calculate the steady-state probability that exactly 1 customer is in the system.

17.5-9) $\lambda = 3$ documents / hour

$\mu_1 = 4$ documents / hour

$\mu_i = 6$ documents / hour for $i \geq 2$

a)



b) $P_1 = \frac{3}{4} P_0$

$P_2 = \frac{3^2}{4 \cdot 6} P_0$

$\vdots$

$P_n = \frac{3^n}{4 \cdot 6^{n-1}} P_0$

$\sum_{n=0}^{\infty} P_n = 1$

$$P_0 = \left[1 + \sum_{n=1}^{\infty} \frac{3^n}{4 \cdot 6^{n-1}}\right]^{-1}$$

$$= \left[1 + \frac{6}{4} \cdot \left(\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - 1\right)\right]^{-1}$$

$$= \left[1 + \frac{3}{2} \underbrace{\left(\frac{1}{1-\frac{1}{2}} - 1\right)}_{=1}\right]^{-1}$$

$$= \frac{2}{5}$$

Then; $P_0 = \frac{2}{5}$; $P_n = \frac{3}{2} \cdot \left(\frac{1}{2}\right)^n$ $n = 1, 2, \ldots$

c) We have; $\rho = \frac{1}{2}$, $P_n = \frac{3}{2} \cdot \rho^n$ $n = 1, 2, \ldots$

$$L = \sum_{n=0}^{\infty} n \cdot P_n = \sum_{n=0}^{\infty} n \cdot \frac{3}{2} \rho^n$$

$$= \frac{3}{2} \rho \cdot \sum_{n=0}^{\infty} n \cdot \rho^{n-1} = \frac{3}{2} \rho \sum_{n=0}^{\infty} \frac{d}{d\rho} \rho^n$$

$$= \frac{3}{2} \rho \cdot \frac{d}{d\rho} \sum_{n=0}^{\infty} \rho^n = \frac{3}{2} \rho \cdot \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right)$$

$$= \frac{3}{2} \cdot \rho \cdot \frac{1}{(1-\rho)^2} = \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{\frac{1}{4}} = 3$$

$$W = \frac{L}{\lambda} = \frac{3}{3} = 1 \text{ hours}; \quad L_q = \sum_{n=1}^{\infty} (n-1) \cdot P_n = \sum_{n=1}^{\infty} n P_n - \sum_{n=1}^{\infty} P_n$$

$$= \underbrace{\sum_{n=0}^{\infty} n P_n}_{=L} - \sum_{n=0}^{\infty} P_n + P_0 = 3 - 1 + \frac{2}{5} = \frac{12}{5} = 2{,}4$$

$$W_q = \frac{L_q}{\lambda} = \frac{2{,}4}{3} = 0{,}8 \text{ hours}$$

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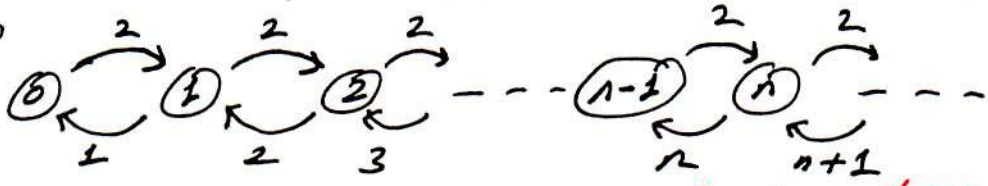
17.5-10) $\lambda = 2$ customers/min. ; $\mu = 1$ customers/min. ; $P_1 = ?$

$s = \infty$



* By min. of exponentials property.

$P_1 = \frac{2}{1} P_0$

$P_2 = \frac{2^2}{1.2} P_0$

$\vdots$

$P_n = \frac{2^n}{n!} \cdot P_0$

$\sum_{n=0}^{\infty} P_n = 1$

$P_0 = \left[\sum_{n=0}^{\infty} \frac{2^n}{n!}\right]^{-1} = (e^2)^{-1} = e^{-2}$

$P_n = \frac{2^n}{n!} \cdot e^{-2}$

Then; $P_2 = 2 \cdot e^{-2} = 0{,}2707$

17.5-11. Suppose that a single-server queueing system fits all the assumptions of the birth-and-death process *except* that customers always arrive in *pairs*. The mean arrival rate is 2 pairs per hour (4 customers per hour) and the mean service rate (when the server is busy) is 5 customers per hour.

(a) Construct the rate diagram for this queueing system.

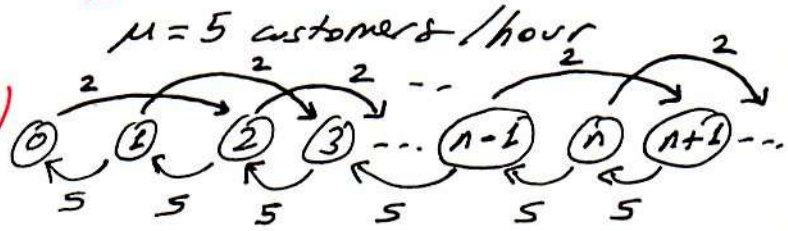
(b) Develop the balance equations.

(c) For comparison purposes, display the rate diagram for the corresponding queueing system that completely fits the birth-and-death process, i.e., where customers arrive *individually* at a mean rate of 4 per hour.

17.5-11) $\lambda = 2$ pairs/hour (4 customers)

$\mu = 5$ customers/hour

a)



b) $2P_0 = 5P_1$

$7P_1 = 5P_2$

$7P_2 = 2P_0 + 5P_3$

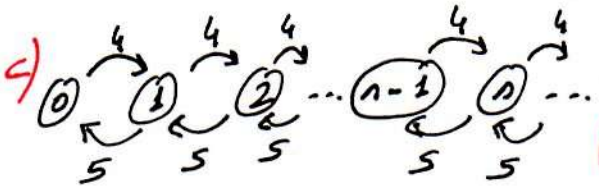
$\vdots$

$7P_n = 2P_{n-2} + 5P_{n+1}$

$\vdots$

$\sum_{n=0}^{\infty} P_n = 1$

c)



$\rho = \frac{4}{5}$; $P_0 = 1 - \rho = \frac{1}{5}$

$P_n = (1-\rho) \cdot \rho^n$

$= \frac{1}{5} \cdot \left(\frac{4}{5}\right)^n$

17.5-13. Consider a queueing system that has two classes of customers, two clerks providing service, and *no queue*. Potential customers from each class arrive according to a Poisson process, with a mean arrival rate of 10 customers per hour for class 1 and 5 customers per hour for class 2, but these arrivals are lost to the system if they cannot immediately enter service.

Each customer of class 1 that enters the system will receive service from either one of the clerks that is free, where the service times have an exponential distribution with a mean of 5 minutes.

Each customer of class 2 that enters the system requires the *simultaneous use of both clerks* (the two clerks work together as a single server), where the service times have an exponential distribution with a mean of 5 minutes. Thus, an arriving customer of this kind would be lost to the system unless both clerks are free to begin service immediately.

(a) Formulate the queueing model as a continuous time Markov chain by defining the states and constructing the rate diagram.

(b) Now describe how the formulation in part (*a*) can be fitted into the format of the birth-and-death process.

(c) Use the results for the birth-and-death process to calculate the steady-state joint distribution of the number of customers of each class in the system.

(d) For each of the two classes of customers, what is the expected fraction of arrivals who are unable to enter the system?

17.5-13) $\lambda_1 = 10$ customers/hour

$\lambda_2 = 5$ customers/hour

$s = 2$ / K = System Capacity $= 2$

$\mu = 12$ customers/hour.

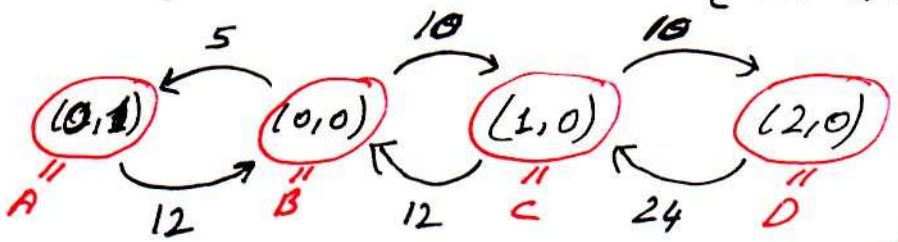
Class 2 Customer $\Rightarrow$ Uses both servers.

a) $\{X_t$: # of type 1 and type 2 customers in the system at time $t\}$

(x,y): (Type 1, Type 2)

$SS = \{(0,1); (0,0); (1,0); (2,0)\}$

b)



c) $12P_A = 5P_B \Rightarrow P_B = 2{,}4P_A$

$15P_B = 12P_A + 12P_C \Rightarrow 36P_A = 12P_A + 12P_C$

$P_C = 2P_A$

$22P_C = 10P_B + 24P_D$ × OMIT

$24P_D = 10P_C \Rightarrow P_D = \frac{10}{24}P_C = \frac{10}{24}\cdot 2P_A = 0{,}833P_A$

$P_A + P_B + P_C + P_D = 1 \rightarrow P_A(1 + 2{,}4 + 2 + 0{,}833) = 1$

$P_A = 0{,}160 \Rightarrow P_B = 0{,}385;\ P_C = 0{,}321;\ P_D = 0{,}134$

Type 1 \ Type 2	0	1
0	0,385	0,160
1	0,321	0
2	0,134	0

Joint pmf

d) Total $= P_A + P_D = 0{,}160 + 0{,}134 = 0{,}294$

Class 1 $= 0{,}294 \cdot \frac{10}{10+5} = 0{,}196$; Class 2 $= 0{,}294 \cdot \frac{5}{10+15} = 0{,}098$

By the corollary of Min. of exponentials.

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