STOCHASTIC MODELS CHAPTERS
LECTURE NOTES 16.6-16.7-16.8

First Passage Times

* $f_{ij}^{(n)}$ is going from state i to state j first time in n steps. Then, we have;

$$f_{ij}^{(1)} = p_{ij}^{(1)} = p_{ij}$$

$$f_{ij}^{(2)} = \sum_{k \neq j} p_{ik} \cdot f_{kj}^{(1)} \quad \text{so,} \quad f_{ij}^{(n)} = \sum_{k \neq j} p_{ik} \cdot f_{kj}^{(n-1)}$$

$\underbrace{k \neq j}$ → Because we want "first"

* Going from state i to itself first time is called recurrence time

* First passage times are random variables.

Ex. Inventory

$X_0 = 3 \quad X_1 = 1 \quad X_2 = 2 \quad X_3 = 1 \quad X_4 = 3 \quad X_5 = 0 \quad X_6 = 3 \ldots$

First passage time from 3 to 1 is 1

from 3 to 0 is 5

Recurrence time of state 3 is 4

Remember;

$$P = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.080 & 0.184 & 0.368 & 0.368 \\ 0.632 & 0.368 & 0 & 0 \\ 0.264 & 0.368 & 0.368 & 0 \\ 0.080 & 0.184 & 0.368 & 0.368 \end{bmatrix} \end{array}$$

WPT we reach from state 3 to state 0 "first time" 3 weeks later?

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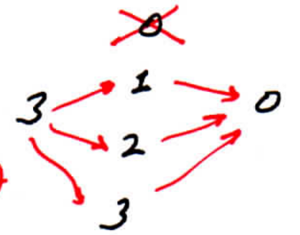
$f_{30}^{(1)} = p_{30} = 0{,}080$

$f_{30}^{(2)} = p_{31} \cdot p_{10} + p_{32}\, p_{20} + p_{33}\, p_{30}$

$= 0{,}184 \cdot 0{,}632 + 0{,}368 \cdot 0{,}264 + 0{,}368 \cdot 0{,}080$

$= 0{,}243$



$f_{30}^{(3)} = p_{31}\, p_{11}\, p_{10} + p_{31}\, p_{12}\, p_{20} + p_{33}\, p_{33}\, p_{30} + \dots + p_{33}\, p_{33}\, p_{31}$

We have; $\sum_{n=1}^{\infty} f_{ij}^{(n)} \le 1$

(i) If i and j are in the same recurrent class, $f_{ij}^{(n)}$ is a probability distribution and $\sum_{n=1}^{\infty} f_{ij}^{(n)} = 1$

(ii) Otherwise, $\sum f_{ij}^{(n)} < 1$

* μ_{ij} is "Expected First Passage Time" from state i to state j. Then;

(i) $\Rightarrow \mu_{ij} = \sum_{n=1}^{\infty} n \cdot f_{ij}^{(n)}$

(ii) $\Rightarrow \mu_{ij} \to \infty$ (May not reach forever)

We find μ_{ij} by the linear equation system;

$$\mu_{ij} = 1 + \sum_{k \ne j} p_{ik}\, \mu_{kj}$$

Ex. Inventory starting with 3 cams, what is the expected number of weeks to go out of stock (first time)?

$\mu_{30} = 1 + p_{31}\,\mu_{10} + p_{32}\,\mu_{20} + p_{33}\,\mu_{30}$

$\mu_{20} = 1 + p_{21}\,\mu_{10} + p_{22}\,\mu_{20} + p_{23}\,\mu_{30}$

$\mu_{10} = 1 + p_{11}\,\mu_{10} + p_{12}\,\mu_{20} + p_{13}\,\mu_{30}$

Then; $\mu_{30} = 1 + 0{,}184\,\mu_{10} + 0{,}368\,\mu_{20} + 0{,}368\,\mu_{30}$

$\mu_{20} = 1 + 0{,}368\,\mu_{10} + 0{,}368\,\mu_{20}$

$\mu_{10} = 1 + 0{,}368\,\mu_{10}$

$\mu_{10} = 1{,}58$; $\mu_{20} = 2{,}51$; $\boxed{\mu_{30} = 3{,}50}$

If we are out of stock, what is the expected number of weeks to go out of stock again?

$\mu_{ii} = \frac{1}{\pi_i} \Rightarrow \mu_{00} = \frac{1}{\pi_0} = \frac{1}{0{,}286} = 3{,}5$ weeks.

16.6-2. A manufacturer has a machine that, when operational at the beginning of a day, has a probability of 0.1 of breaking down sometime during the day. When this happens, the repair is done the next day and completed at the end of that day.

(a) Formulate the evolution of the status of the machine as a Markov chain by identifying three possible states at the end of each day, and then constructing the (one-step) transition matrix.

(b) Use the approach described in Sec. 16.6 to find the μ_{ij} (the expected first passage time from state i to state j) for all i and j. Use these results to identify the expected number of full days that the machine will remain operational before the next breakdown after a repair is completed.

(c) Now suppose that the machine already has gone 20 full days without a breakdown since the last repair was completed. How does the expected number of full days *hereafter* that the machine will remain operational before the next breakdown compare with the corresponding result from part (*b*) when the repair had just been completed? Explain.

16.6-3. Reconsider Prob. 16.6-2. Now suppose that the manufacturer keeps a spare machine that only is used when the primary machine is being repaired. During a repair day, the spare machine has a probability of 0.1 of breaking down, in which case it is repaired the next day. Denote the state of the system by (x, y), where x and y, respectively, take on the values 1 or 0 depending upon whether the primary machine (x) and the spare machine (y) are operational (value of 1) or not operational (value of 0) at the end of the day. [*Hint*: Note that (0, 0) is not a possible state.]

(a) Construct the (one-step) transition matrix for this Markov chain.

(b) Find the *expected recurrence time* for the state (1, 0).

16.6-2) $P\{\text{Break down}\} = 0{,}1$

a) $\{X_n$: state of the machine at the end of the day, $n \in N\}$

$SS = \{0, 1, 2\}$

0: Mach. was NOT broken during the day

1: Mach. was broken during the day

2: Mach was under repair during the day

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0{,}9 & 0{,}1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

b) $\mu_{01} = 1 + 0{,}9 \cdot \mu_{01} \Rightarrow \boxed{\mu_{01} = 10}$

$\mu_{10} = 1 + \mu_{20} \Rightarrow \mu_{10} = 2$

$\mu_{20} = 1$

c) By Markovian property,

μ_{10} / working already 20 days

$= 20 + 10 = \underline{30}$

16.6-3) (x,y): (Primary mach, Spare Mach) 1: Operational
0: Broken down

a)

$$P = \begin{array}{c} \\ A:(1,1) \\ B:(0,1) \\ C:(1,0) \end{array} \begin{array}{c} \begin{array}{ccc} A:(1,1) & B:(0,1) & C:(1,0) \end{array} \\ \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.9 & 0 & 0.1 \\ 0.9 & 0.1 & 0 \end{bmatrix} \end{array}$$

where $\{X_n$: States of the machines at the end of day n, $n \in \mathbb{N}\}$

$SS = \{(1,1), (0,1), (1,0)\}$

b) $\pi_A = 0.9\pi_A + 0.9\pi_B + 0.9\pi_C$ X OMIT

① $\pi_B = 0.1\pi_A + \quad 0.1\pi_C$

② $\pi_C = \cancel{9} \quad 0.1\pi_B$

③ $\pi_A + \pi_B + \pi_C = 1$

① $\Rightarrow \pi_B = 0.1\pi_A + 0.1 \cdot 0.1\pi_B \Rightarrow 0.1\pi_A = 0.99\pi_B$
$\pi_A = 9.9\pi_B$

③ $\Rightarrow 9.9\pi_B + \pi_B + 0.1\pi_B = 1$

$\pi_B = \frac{10}{110}$; $\pi_C = \frac{1}{110}$; $\pi_A = \frac{99}{110}$

Then; $\mu_{(1,0);(1,0)} = \mu_{CC} = \frac{1}{\pi_C} = \underline{\underline{110}}$

Other recurrence times are;

$\mu_{BB} = \frac{1}{\pi_B} = 11$ and $\mu_{AA} = \frac{1}{\pi_A} = \frac{110}{99} \cong 1.111$

(Comment on the results)

Absorbing States

Let, k is an absorbing state. Then;

$$f_{ik} = \sum_{j=0}^{M} P_{ij} \cdot f_{jk}$$

we have, $f_{kk} = 1$ and if r is ANOTHER absorbing state, or r is a recurrent state $r \neq k$ then $f_{rk} = 0$

Ex. Gambler's Ruin

Let $P\{\text{You win a game}\} = p = \frac{3}{4}$ and $SS = \{0, 1, 2, 3\}$ for simplicity (i.e, you and your opponent have a total of \$3) starting with 2\$, WPT you loose the game?

Ans//

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$f_{20} = ?$

$$f_{00} = 1$$

$$f_{10} = P_{10} \underbrace{f_{00}}_{=1} + \underbrace{P_{11}}_{=0} f_{10} + P_{12} f_{20} + \underbrace{P_{13}}_{=0} f_{30}$$

$$f_{20} = \underbrace{P_{20}}_{=0} f_{00} + P_{21} f_{10} + \underbrace{P_{22}}_{=0} f_{20} + P_{23} \underbrace{f_{30}}_{=0}$$

$$f_{30} = 0$$

$$f_{10} = \frac{1}{4} + \frac{3}{4} f_{20}$$

$$f_{20} = \frac{1}{4} f_{10}$$

$f_{10} = 0{,}308$ $\boxed{f_{20} = 0{,}077}$

* Note that $f_{23} = 1 - f_{20} = 0{,}923$ and $f_{13} = 1 - f_{10} = 0{,}692$ since you either win or loose the game.

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16.7-2. A video cassette recorder manufacturer is so certain of its quality control that it is offering a complete replacement warranty if a recorder fails within 2 years. Based upon compiled data, the company has noted that only 1 percent of its recorders fail during the first year, whereas 5 percent of the recorders that survive the first year will fail during the second year. The warranty does not cover replacement recorders.

(a) Formulate the evolution of the status of a recorder as a Markov chain whose states include two absorption states that involve needing to honor the warranty or having the recorder survive the warranty period. Then construct the (one-step) transition matrix.

(b) Use the approach described in Sec. 16.7 to find the probability that the manufacturer will have to honor the warranty.

16.7-2) a) $\{X_n : \text{Status of the recorder in its } n^{th} \text{ year}, n \in \mathbb{N}^+\}$

$X_1 = 1$

$SS = \{0, 1, 2, 3\}$

0: Out of warranty by replacement

1: Recorder is in its first year

2: Survived to second year

3: Out of warranty by surviving 3rd year.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0{,}01 & 0 & 0{,}99 & 0 \\ 0{,}05 & 0 & 0 & 0{,}95 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

b) $f_{10} = ?$

$f_{00} = 1$

$f_{10} = 0{,}01 \cdot f_{00} + 0{,}99 \cdot f_{20}$

$f_{20} = 0{,}05 \cdot f_{00} + 0{,}95 \cdot f_{30}$

$f_{30} = 0$

$f_{10} = 0{,}01 \cdot 0{,}99 \cdot f_{20}$

$f_{20} = 0{,}05$

$f_{10} = 0{,}01 + 0{,}99 \cdot 0{,}05 = 0{,}0595$

c) If the recorders are produced by 50\$ and the sales price is 80\$, what is the average profit per recorder?

Ans: $f_{13} = 1 - f_{10}$

$= 1 - 0{,}0505 = 0{,}9405$

Profit	30	-20
P(Profit)	0,9405	0,0505

$E(\text{Profit}) = \sum p_i \cdot P(p_i)$

$= 30 \cdot 0{,}9405 - 20 \cdot 0{,}0505$

$= 27{,}205\ \$$

Continuous Time Markov Chains

$\{X(t'); t' \geq 0\}$ is a Continuous time Markov chain if it has the Markovian property;

$$P\{X(t+s)=j \mid X(s)=i \text{ and } X(r)=x(r)\} = P\{X(t+s)=j \mid X(s)=i\}$$

where $s>r$

$SS = \{0, 1, 2, \ldots, M\}$

The only distribution that has the Markovian property is the exponential distribution.

Since Markov chain is continuous, we do NOT have one step transition probabilities (there's NO step). Instead, we have "rate diagram", which shows the rates q_{ij} : rate from state i to state j

The steady state equations are found by;

$$\pi_j q_j = \sum_{i \neq j} \pi_i q_{ij} \quad \text{for } j = 0, 1, \ldots, M$$

$$\sum \pi_j = 1$$

where $q_i = \sum_{j \neq i} q_{ij}$.

Steady state equations of CTMC are called "balance equations". We'll see them in detail at "Queueing theory" chapter. So, here is just a summary.

Ex// A shop has two machines which breaks down (each) with exponential with a mean of 1 day. A single repairman repairs them with exponential with a mean of $\frac{1}{2}$ day. Find the steady state distribution of number of machines broken.

Ans// Note that; $MEAN = \frac{1}{RATE}$ because Rate means # of events per unit time.

Then, Repair Rate $= \frac{1}{1/2} = 2$ machines / day

Broken Rate $= \frac{1}{1} = 1$ machines / day (or a machine is broken down 1 times / day)

Also note that, if two machines are working, their (TOTAL) rate of brake-down is $1+1=2$.

The rate diagram is;

State
$$\begin{tikzcd}
0 \arrow[r, bend left, "q_{01}=2 \to 1+1"] & 1 \arrow[l, bend left, "q_{10}=2"] \arrow[r, bend left, "q_{12}=1"] & 2 \arrow[l, bend left, "q_{21}=2"]
\end{tikzcd}$$

where $\{X(t): \#$ of machines broken down at time $t \geq 0\}$

Then; $2\pi_0 = 2\pi_1$

$3\pi_0 = 2\pi_0 + 2\pi_2$

$2\pi_2 = \pi_1$

$\pi_0 + \pi_1 + \pi_2 = 1$

$\pi_0 = \frac{2}{5}$; $\pi_1 = \frac{2}{5}$; $\pi_2 = \frac{1}{5}$

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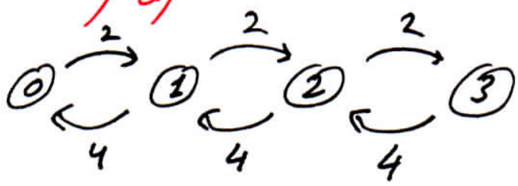
16.8-1. Reconsider the example presented at the end of Sec. 16.8. Suppose now that a third machine, identical to the first two, has been added to the shop. The one maintenance person still must maintain all the machines.

(a) Develop the *rate diagram* for this Markov chain.
(b) Construct the *steady-state equations.*
(c) Solve these equations for the *steady-state probabilities.*

16.8-2. The state of a particular continuous time Markov chain is defined as the number of jobs currently at a certain work center, where a maximum of three jobs are allowed. Jobs arrive individually. Whenever fewer than three jobs are present, the time until the next arrival has an exponential distribution with a mean of $\frac{1}{2}$ day. Jobs are processed at the work center one at a time and then leave immediately. Processing times have an exponential distribution with a mean of $\frac{1}{4}$ day.

(a) Construct the *rate diagram* for this Markov chain.
(b) Write the *steady-state equations.*
(c) Solve these equations for the *steady-state probabilities.*

16.8-2) a)

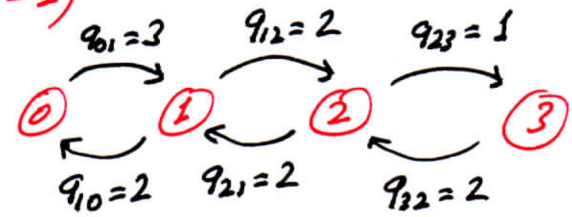


b)
$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$
$$2\pi_0 = 4\pi_1$$
$$6\pi_1 = 2\pi_0 + 4\pi_2$$
$$6\pi_2 = 2\pi_1 + 4\pi_3$$
$$2\pi_2 = 4\pi_3$$

c)
$$\pi_1 = \frac{2}{4}\pi_0 = 0,2667$$
$$\pi_2 = \frac{2.2}{4.4}\pi_0 = 0,1333$$
$$\pi_3 = \frac{2.2.2}{4.4.4}\pi_0 = 0,0667$$
$$\pi_0\left[1 + \frac{2}{4} + \frac{2.2}{4.4} + \frac{2.2.2}{4.4.4}\right] = 1$$
$$\pi_0 = 0,5333$$

16.8-1)

a)



b)
① $3\pi_0 = 2\pi_1$

② $4\pi_1 = 3\pi_0 + 2\pi_2$

$3\pi_2 = 2\pi_1 + 2\pi_3$ × OMIT

③ $\pi_2 = 2\pi_3$

④ $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

① ⇒ $\pi_1 = \frac{3}{2}\pi_0 = 0,3158$

② ⇒ $4.\frac{3}{2}\pi_0 - 3\pi_0 = 2\pi_2$

$\pi_2 = \frac{3}{2}\pi_0 = 0,3158$

③ ⇒ $2\pi_3 = \frac{3}{2}\pi_0$

$\pi_3 = \frac{3}{4}\pi_0 = 0,1579$

④ ⇒ $\pi_0\left[1 + \frac{3}{2} + \frac{3}{2} + \frac{3}{4}\right] = 1$

$\pi_0 = \frac{1}{4,75} = 0,2105$

n	0	1	2	3
π_n	0,2105	0,3158	0,3158	0,1579