LECTURE NOTES PROBABILITY — CHAPTER 5

MULTIVARIATE PROBABILITY DISTRIBUTIONS

Discrete Case

5.3 The same study that produced the seat belt safety data of Table 5.1 also took into account the age of the child involved in a fatal accident. The results for those children wearing *no* seat belts are shown below. Here, $X_1 = 1$ if the child did not survive and X_2 indicates the age in years. (An age of zero implies that the child was less than 1 year old, an age of 1 implies the child was more than 1 year old but not yet 2, and so on.)

Age	Survivors	Fatalities
0	104	127
1	165	91
2	267	107
3	277	90
4	316	94

a Construct an approximate joint probability distribution for X_1 and X_2.

b Construct the conditional distribution of X_1 for fixed values of X_2. Discuss the implications of these results.

c Construct the conditional distribution of X_2 for fixed values of X_1. Are the implications the same as in part (b)?

5.3) a)

X_1: Result

X_2: Age	0: Survivor	1: Fatality	TOTAL
0	$\frac{104}{1693} = 0,063$	$\frac{127}{1693} = 0,078$	231
1	$\frac{165}{1693} = 0,101$	$\frac{91}{1693} = 0,056$	256
2	$\frac{267}{1693} = 0,163$	$\frac{107}{1693} = 0,065$	374
3	$\frac{277}{1693} = 0,169$	$\frac{90}{1693} = 0,055$	367
4	$\frac{316}{1693} = 0,193$	$\frac{94}{1693} = 0,057$	410
TOTAL	1129	509	1639

* Remember, for events A and B, the joint probability is $P(A \cap B)$ or in short, $P(AB)$.

The joint probability distribution of random variables X_1 and X_2 is;

$$f(x_1, x_2) = P(X_1 = x_1, X_2 = x_2)$$

This is the probability of $X_1 = x_1$ AND $X_2 = x_2$.

For example, a randomly chosen accident has age 3 AND fatal is the ratio of 107 to all accidents: 1639. Then,

$$P(X_1 = 1, X_2 = 3) = \frac{107}{1639} = 0{,}065$$

* For $f(x_1, x_2)$ to be a joint pmf of random variables X_1 and X_2, we have

(i) $f(x_1, x_2) \geq 0$

(ii) $\sum_{x_1} \sum_{x_2} f(x_1, x_2) = 1$

* Remember, single event probabilities are called Marginal probabilities. From total probability rule, for events A and B, we have;

$$P(A) = P(AB) + P(A\bar{B})$$

The marginal pmf's $f_1(x_1)$ and $f_2(x_2)$ are found by summing all possible values of the other variable. Namely;

$$f_1(x_1) = \sum_{x_2} f(x_1, x_2) \quad \text{and}$$

$$f_2(x_2) = \sum_{x_1} f(x_1, x_2)$$

To illustrate, if we are NOT interested in age of the children but interested in if the child survived,

$$P(\text{Survive}) = \frac{1129}{1639} = \frac{106+165+267+277+316}{1639} = 0{,}689$$

So; $P(\text{Fatality}) = 1 - 0{,}689 = 0{,}311$.

This corresponds to;

$$P(X_1=0) = \sum_{j=0}^{4} P(X_1=0, X_2=j) = \sum_{j=0}^{4} f(0,j)$$

$$= 0{,}063 + 0{,}101 + 0{,}163 + 0{,}169 + 0{,}193 = 0{,}689.$$

X_2 \ X_1	0	1	TOTAL
0	0,063	0,078	0,141
1	0,101	0,056	0,157
2	0,163	0,065	0,228
3	0,169	0,055	0,224
4	0,193	0,057	0,250
TOTAL	0,689	0,311	1,000

So, the marginal distributions are;

x_1	0	1
$f_1(x_1)$	0,689	0,311

x_2	0	1	2	3	4
$f_2(x_2)$	0,141	0,157	0,228	0,224	0,250

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b) Remember, the conditional probability $P(A|B)$ stands for the probability of A given that we know event B had occured. Likewise,

$$f_{1|2}(x_1 | X_2 = x_2) = P(X_1 = x_1 | X_2 = x_2)$$

is the conditional probability of $X_1 = x_1$ given that $X_2 = x_2$ is fixed (known to be occured).

For example, if we know that the child is 3 years old, the probability to survive is;

$$P(X_1 = 0 | X_2 = 3) = f_{1|2}(0 | X_2 = 3) = \frac{277}{367} = 0,755$$

So, 277 out of 367 children whose age is 3 is survived. This corresponds to 75,5% chance.

Observe that;

$$P(X_1 = 0 | X_2 = 3) = \frac{277}{367} = \frac{277/1639}{367/1639} = \frac{P(X_1 = 0, X_2 = 3)}{P(X_2 = 3)}$$

So, in general;

$$f_{1|2}(x_1 | X_2 = x_2) = \frac{f(x_1, x_2)}{f_2(x_2)}$$

Note that, $f_{1|2}(x_1 | X_2 = x_2)$ is a function of x_1 because we fix X_2 to x_2.

The following table shows conditional probability distribution of X_1 for each fixed value of $X_2 = x_2$.

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x_1	0	1
$f_{1\|2}(x_1 \| X_2=0)$	$\frac{0{,}063}{0{,}141} = 0{,}45$	$\frac{0{,}078}{0{,}141} = 0{,}55$
$f_{1\|2}(x_1 \| X_2=1)$	$\frac{0{,}101}{0{,}157} = 0{,}64$	$\frac{0{,}056}{0{,}157} = 0{,}36$
$f_{1\|2}(x_1 \| X_2=2)$	$\frac{0{,}163}{0{,}228} = 0{,}71$	$\frac{0{,}065}{0{,}228} = 0{,}29$
$f_{1\|2}(x_1 \| X_2=3)$	$\frac{0{,}169}{0{,}224} = 0{,}75$	$\frac{0{,}055}{0{,}224} = 0{,}25$
$f_{1\|2}(x_1 \| X_2=4)$	$\frac{0{,}193}{0{,}250} = 0{,}77$	$\frac{0{,}057}{0{,}250} = 0{,}23$

Observe that the row totals are 1.

$f_{1|2}(x_1|X_2=x_2)$ is a conditional pmf if

$$f_{1|2}(x_1|X_2=x_2) \geq 0$$

$$\sum_{x_1} f_{1|2}(x_1|X_2=x_2) = 1$$

c) The age distribution of survivors is;

x_2	0	1	2	3	4
$f_{2\|1}(x_2\|X_1=0)$	$\frac{0{,}063}{0{,}689} = 0{,}092$	$\frac{0{,}101}{0{,}689} = 0{,}146$	$\frac{0{,}163}{0{,}689} = 0{,}236$	$\frac{0{,}169}{0{,}689} = 0{,}245$	$\frac{0{,}193}{0{,}689} = 0{,}280$

And the fatalities is;

x_2	0	1	2	3	4
$f_{2\|1}(x_2\|X_1=1$	$\frac{0{,}078}{0{,}311} = 0{,}250$	$\frac{0{,}056}{0{,}311} = 0{,}179$	$\frac{0{,}065}{0{,}311} = 0{,}210$	$\frac{0{,}055}{0{,}311} = 0{,}178$	$\frac{0{,}057}{0{,}311} = 0{,}185$

* Remember, the expected value (mean, average) of a discrete random variable is

$$\mu = \sum_x x \cdot f(x)$$

Then, for our example, the mean age of the children under study is;

$$\mu_2 = \sum_{x_2=0}^{4} x_2 \cdot f_2(x_2) = 0 \cdot 0{,}161 + 1 \cdot 0{,}157 + \dots + 4 \cdot 0{,}250$$

$$\mu_2 = 2{,}285$$

The mean percentage of fatals is;

$$\mu_1 = \sum_{x_1=0}^{1} x_1 \cdot f_1(x_1) = 0 \cdot 0{,}689 + 1 \cdot 0{,}311 = 0{,}311 .$$

Also, we can find the mean age of the children who survived;

$$E(X_2 \mid X_1 = 0) = \sum_{x_2=0}^{4} x_2 \cdot f_{2|1}(x_2 \mid X_1 = 0)$$

$$= 0 \cdot 0{,}092 + 1 \cdot 0{,}146 + \dots + 4 \cdot 0{,}280 = 2{,}473$$

And, for example, the mean percentage of fatals whose age is 4 is;

$$E(X_1 \mid X_2 = 4) = \sum_{x_1=0}^{1} x_1 \cdot f_{1|2}(x_1 \mid X_2 = 4)$$

$$= 0 \cdot 0{,}77 + 1 \cdot 0{,}23 = 0{,}23$$

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* Remember, two events A and B are independent if the occurance of one event does NOT change the probability of the other event. Then,

$$P(A|B) = P(A)$$

$$\frac{P(AB)}{P(B)} = P(A)$$

$$P(AB) = P(A).P(B) \text{ iff A and B are independent.}$$

Likewise, two random variables are independent if and only if every possible choice of x_1 and x_2 are independent.

X_1 and X_2 are independent iff

$$f(x_1, x_2) = f_1(x_1).\ f_2(x_2) \quad \forall x_1, x_2$$

To see if Fatality and Age are independent,

$$f(0,3) \stackrel{?}{=} f_1(0).\ f_2(3)$$

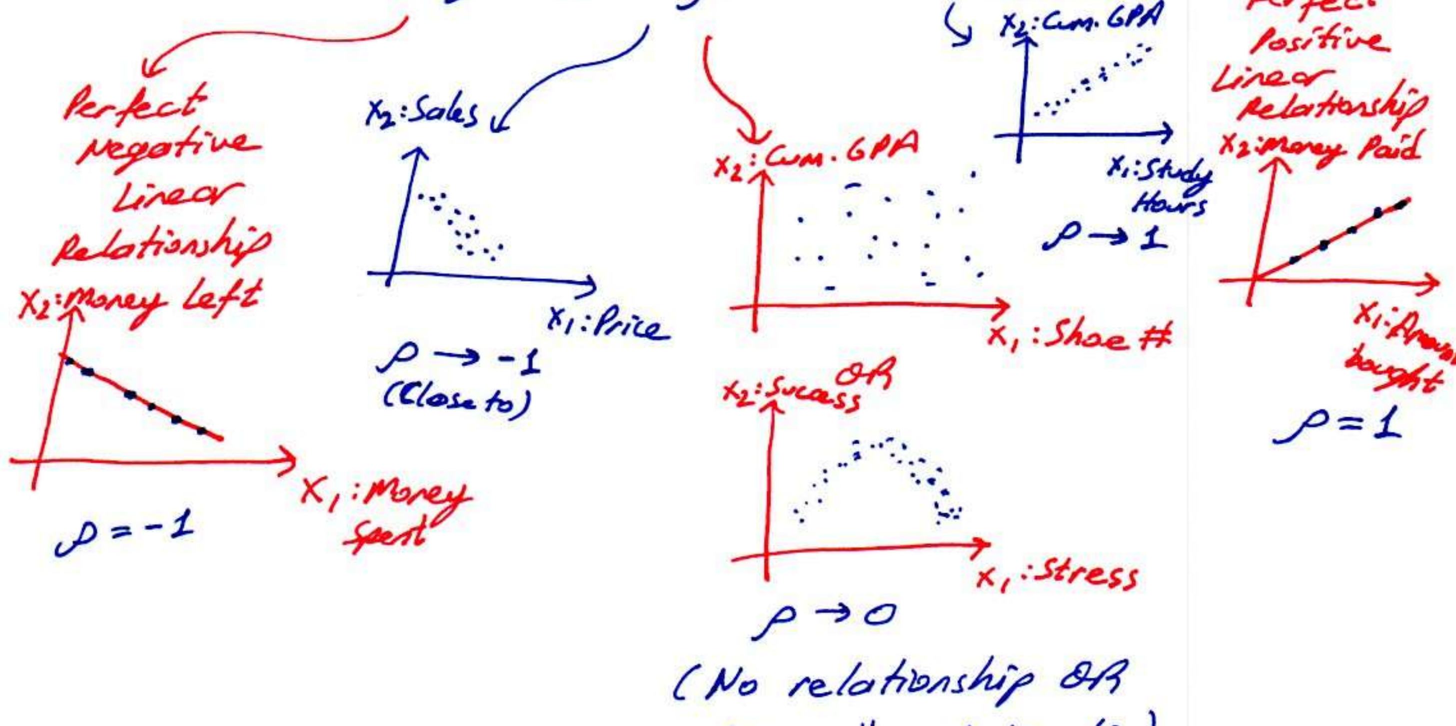
$$0{,}169 \stackrel{?}{=} 0{,}689 \cdot 0{,}224$$

$$0{,}169 \neq 0{,}154$$

Therefore, Fatality and Age are NOT independent. It makes sense because we may think that older children have more probability to survive.

* Surely, perfect independence rarely occurs. To measure "Linearly Dependence", we use "ρ: Correlation Coefficient". ρ is a number between -1 and 1, which shows how strong and in what direction the linear relationship between variables X_1 and X_2 is. To illustrate,

$$-1 \le \rho \le 1$$



(No relationship OR No "linear" relationship)

Note that, if price increases, sales is expected to decrease linearly, but not a perfect relationship because there are other factors effecting Sales. Same reasoning is between Studying Hours and Cum. GPA, but in a positive way.

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Also Note that:

INDEPENDENCE $\Rightarrow$ UNCORRELATION

But the converse is NOT true as a fact. If two random variables are Uncorrelated, it means that "There is No Linear relationship between them." But, as in Stress and Success example, there may be some other relationship, for example Quadratic Relationship.

* To calculate correlation coefficient ρ between X_1 and X_2, we follow these steps;

(i) $E(X_1) = \sum_{x_1} x_1 \cdot f_1(x_1)$ $\quad E(X_1^2) = \sum_{x_1} x_1^2 f_1(x_1)$

$E(X_2) = \sum_{x_2} x_2 f_2(x_2)$ $\quad E(X_2^2) = \sum_{x_2} x_2^2 f_2(x_2)$

$$E(X_1 X_2) = \sum_{x_1} \sum_{x_2} x_1 \cdot x_2 \cdot f(x_1, x_2)$$

(ii) $Var(X_1) = E(X_1^2) - E^2(X_1)$

$Var(X_2) = E(X_2^2) - E^2(X_2)$

$Cov(X_1, X_2) = E(X_1 X_2) - E(X_1) E(X_2)$

(iii) $$\rho = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1) \cdot Var(X_2)}}$$

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Ex4 Calculate the correlation coefficient between Age and Fatality in accidents.

Ans4
(i) $E(X_1) = 0,311$; $E(X_2) = 2,285$

$E(X_1^2) = 0^2 \cdot 0,659 + 1^2 \cdot 0,311 = 0,311$

$E(X_2^2) = 0^2 \cdot 0,161 + 1^2 \cdot 0,157 + \cdots + 4^2 \cdot 0,250 = 5,581$

$E(X_1 X_2) = 0 \cdot 0 \cdot 0,063 + 0 \cdot 1 \cdot 0,078 + \cdots + 4 \cdot 1 \cdot 0,057$
$= 0,579$

(ii) $Var(X_1) = 0,311 - 0,311^2 = 0,214$

$Var(X_2) = 5,581 - 2,285^2 = 0,360$

$Cov(X_1, X_2) = 0,579 - 0,311 \cdot 2,285 = -0,132$

(iii) $$\rho = \frac{-0,132}{\sqrt{0,214 \cdot 0,360}} = -0,474$$

There is a negative linear relationship between Age and Fatality in accidents. The relationship is moderately strong (ρ close to 0,5)

Note that Covariance between X_1 and X_2 is negative. We have; $-\infty < Cov(X_1, X_2) < \infty$

and $Var(X_i) \geq 0$ and $Cov(X_1, X_1) = Var(X_1)$
$Cov(X_2, X_2) = Var(X_2)$

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* Mean and Variance of Linear combinations of random variables are found as follows.

Let, $W = aX + bY + c$. Then,

$$E(W) = E(aX+bY+c) = aE(X) + bE(Y) + c$$

$$Var(W) = Var(aX+bY+c) = a^2 Var(X) + b^2 Var(Y) + 2ab\,Cov(X,Y)$$

Note that, if X, Y independent, Cov term is 0.

In general, if $W = \sum a_i X_i$ and $V = \sum b_i Y_i$

$$E(W) = \sum a_i E(X_i)$$

$$Var(W) = \sum a_i^2 Var(X_i) + 2\sum_{i<j}\sum a_i a_j Cov(X_i, X_j)$$

$$Cov(W, V) = \sum_i \sum_j a_i b_j Cov(X_i, Y_j)$$

Continuous Case.

All formulas obtained are valid for continuous Random Variables except that we replace $\sum$ by $\int$. There's a little trick in some questions if X_1's values are conditioned on X_2, or vice-versa. Here, integral bounds should be set carefully. We'll see the details in examples.

5.9 Let X_1 and X_2 denote the proportions of time, out of one workweek, that employees I and II, respectively, actually spend performing their assigned tasks. The joint relative frequency behavior of X_1 and X_2 is modeled by the probability density function

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 & 0 \le x_1 \le 1, 0 \le x_2 \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

a Find $P(X_1 < 1/2, X_2 > 1/4)$.

b Find $P(X_1 + X_2 \le 1)$.

c Are X_1 and X_2 independent?

5.10 Refer to Exercise 5.9. Find the probability that employee I spends more than 75% of the week on her assigned task, given that employee II spends exactly 50% of the workweek on his assigned task.

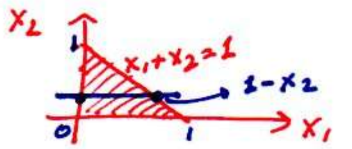
5.9) a) $P(X_1 < 1/2,\ X_2 > 1/4) = \int_{1/4}^{1}\int_{0}^{1/2} f(x_1,x_2)\,dx_1\,dx_2$

$$= \int_{1/4}^{1}\int_{0}^{1/2} (x_1+x_2)\,dx_1\,dx_2 = \int_{1/4}^{1}\left[\frac{x_1^2}{2}+x_2.x_1\right]_{x_1=0}^{x_1=1/2} dx_2$$

$$= \int_{1/4}^{1}\left(\frac{1}{8}+\frac{1}{2}x_2\right)dx_2 = \left(\frac{1}{8}x_2+\frac{1}{2}\frac{x_2^2}{2}\right]_{x_2=1/4}^{x_2=1}$$

$$= \left(\frac{1}{8}+\frac{1}{4}\right)-\left(\frac{1}{8}\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{16}\right) = 0{,}328$$

b) $P(X_1+X_2 \le 1) = P(X_1 \le 1-X_2) = \int_0^1\int_0^{1-x_2} f(x_1,x_2)\,dx_1\,dx_2$



$$= \int_0^1\int_0^{1-x_2}(x_1+x_2)\,dx_1\,dx_2$$

$$= \int_0^1\left[\frac{x_1^2}{2}+x_2x_1\right]_{x_1=0}^{x_1=1-x_2}dx_2 = \int_0^1 \frac{(1-x_2)^2}{2}+(1-x_2).x_2\,dx_2$$

$$= \int_0^1\left[\frac{1}{2}(1-2x_2+x_2^2)+x_2-x_2^2\right]dx_2 = \frac{1}{2}\left(x_2-x_2^2+\frac{x_2^3}{3}\right)\Big|_0^1+\left(\frac{x_2^2}{2}-\frac{x_2^3}{3}\right)\Big|_0^1=\frac{1}{3}$$

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c) $f_1(x_1) = \int_0^1 f(x_1, x_2)\,dx_2 = \int_0^1 (x_1 + x_2)\,dx_2$

$= \left[x_1 x_2 + \frac{x_2^2}{2} \right]_{x_2=0}^{x_2=1} = x_1 + \frac{1}{2} \qquad f_1(x_1) = \begin{cases} x_1 + \frac{1}{2} & 0 \le x_1 \le 1 \\ 0 & \text{o.w.} \end{cases}$

$f_2(x_2) = \int_0^1 f(x_1, x_2)\,dx_1 = \int_0^1 (x_1 + x_2)\,dx_1$

$= \left[\frac{x_1^2}{2} + x_2 x_1 \right]_{x_1=0}^{x_1=1} = x_2 + \frac{1}{2} \qquad f_2(x_2) = \begin{cases} x_2 + \frac{1}{2} & 0 \le x_1 \le 2 \\ 0 & \text{o.w.} \end{cases}$

$f(x_1, x_2) \neq f_1(x_1) . f_2(x_2)$

So, X_1 and X_2 are NOT independent.

5.10) $P(X_1 > 0{,}75 \mid X_2 = 0{,}50) = \frac{P(X_1 > 0{,}75, X_2 = 0{,}50)}{f_2(x_2 = 0{,}50)}$

$= \frac{\int_{0{,}75}^{1} f(x_1; 0{,}50)\,dx_1}{f_2(0{,}50)} = \frac{\left(\frac{x_1^2}{2} + 0{,}50x_1\right]_{x_1=0{,}75}^{x_1=1}}{0{,}50 + 0{,}50} = \frac{\frac{1^2}{2} + 0{,}50 - \left(\frac{0{,}75^2}{2} + 0{,}50 \cdot 0{,}75\right)}{1}$

$= 1 - 0{,}9375 = \underline{\underline{0{,}0625}}$

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5.4 An environmental engineer measures the amount (by weight) of particulate pollution in air samples (of a certain volume) collected over the smokestack of a coal-operated power plant. Let X_1 denote the amount of pollutant per sample when a certain cleaning device on the stack is not operating, and let X_2 denote the amount of pollutant per sample when the cleaning device is operating, under similar environmental conditions. It is observed that X_1 is always greater than $2X_2$, and the relative frequency behavior of (X_1, X_2) can be modeled by

$$f(x_1, x_2) = \begin{cases} k & 0 \le x_1 \le 2, 0 \le x_2 \le 1, 2x_2 \le x_1 \\ 0 & \text{elsewhere} \end{cases}$$

(That is, X_1 and X_2 are randomly distributed over the region inside the triangle bounded by $x_1 = 2$, $x_2 = 0$, and $2x_2 = x_1$.)

a Find the value of k that makes this a probability density function.

b Find $P(X_1 \ge 3X_2)$. (That is, find the probability that the cleaning device will reduce the amount of pollutant by one-third or more.)

5.5 Refer to Exercise 5.4.

a Find the marginal density function for X_1.

b Find $P(X_1 \le 0.5)$.

~~**c** Are X_1 and X_2 independent?~~

5.6 Refer to Exercise 5.4.

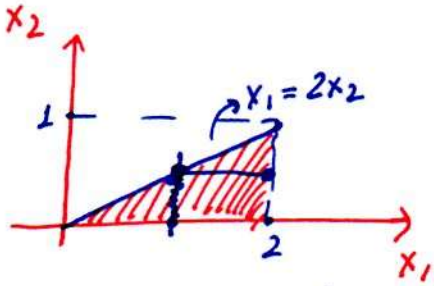
a Find the marginal density function of X_2.

b Find $P(X_2 \le 0.4)$.

c Are X_1 and X_2 independent?

d Find $P(X_2 \le 1/4 | X_1 = 1)$.

5.4) a)



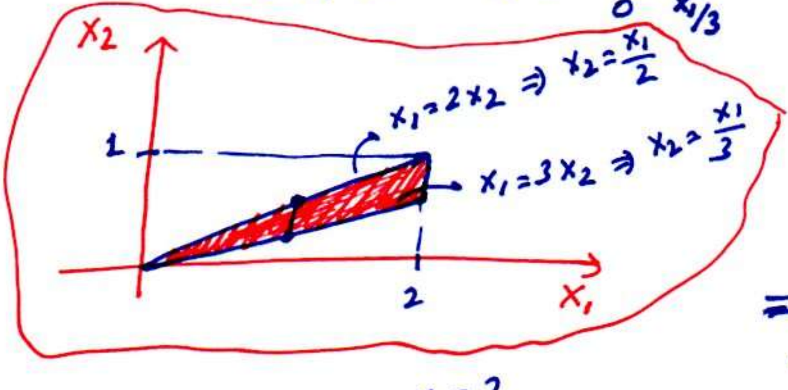
$$\int_0^1 \int_{2x_2}^2 f(x_1, x_2)\, dx_1\, dx_2 = 1$$

$$\int_0^1 \int_{2x_2}^2 f(x_1, x_2)\, dx_1\, dx_2 = \int_0^1 \int_{2x_2}^2 k\, dx_1\, dx_2 = k \int_0^1 \left(x_1 \Big]_{x_1 = 2x_2}^{x_1 = 2} \right) dx_2$$

$$= k \cdot \int_0^1 (2 - 2x_2)\, dx_2 = k \cdot \left[2x_2 - x_2^2 \right]_{x_2 = 0}^{x_2 = 1} = k \cdot (2 - 1) = k = 1$$

$$f(x_1, x_2) = \begin{cases} 1 & 0 \le x_1 \le 2, 0 \le x_2 \le 1, 2x_2 \le x_1 \\ 0 & \text{o.w.} \end{cases}$$

b) $P(X_1 \geq 3X_2) = \int_0^2 \int_{x_1/3}^{x_1/2} f(x_1, x_2)\, dx_2\, dx_1$



$$= \int_0^2 \left(x_2 \Big]_{x_2 = x_1/3}^{x_2 = x_1/2} \right) dx_1$$

$$= \int_0^2 \left(\frac{x_1}{2} - \frac{x_1}{3} \right) dx_1$$

$$= \left(\frac{x_1^2}{4} - \frac{x_1^2}{6} \right]_{x_1=0}^{x_1=2} = \left(\frac{2^2}{4} - \frac{2^2}{6} \right) = 1 - \frac{1}{3} = \frac{2}{3}$$

5.5) a) $f_1(x_1) = \int_0^{x_1/2} f(x_1, x_2)\, dx_2 = \int_0^{x_1/2} dx_2 = x_2 \Big]_{x_2=0}^{x_2 = x_1/2} = \frac{x_1}{2}$

$$f_1(x_1) = \frac{x_1}{2} \qquad 0 \leq x_1 \leq 2$$

b) $P(X_1 \leq 0{,}5) = \int_0^{0{,}5} f_1(x_1)\, dx_1 = \int_0^{0{,}5} \frac{x_1}{2}\, dx_1 = \frac{x_1^2}{4} \Big]_0^{0{,}5} = \frac{0{,}5^2}{4} = \frac{1}{16}$

5.6) a) $f_2(x_2) = \int_{2x_2}^{2} f(x_1, x_2)\, dx_1 = \int_{2x_2}^{2} dx_1 = x_1 \Big]_{x_1 = 2x_2}^{x_1 = 2} = 2 - 2x_2$

$$f_2(x_2) = \begin{cases} 2 - 2x_2 & 0 \leq x_2 \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

b) $P(X_2 \leq 0{,}4) = \int_0^{0{,}4} f_2(x_2)\,dx_2 = \int_0^{0{,}4} (2 - 2x_2)\,dx_2$

$= (2x_2 - x_2^2)\Big]_0^{0{,}4} = 2 \cdot 0{,}4 - 0{,}4^2 = 0{,}64$

c) $f_1(x_1) \cdot f_2(x_2) = \frac{x_1}{2} \cdot (2 - 2x_2) = x_1 \cdot (1 - x_2) \neq f(x_1, x_2) = 1$

X_1 and X_2 are NOT independent

d) $P(X_2 \leq 1/4 \mid X_1 = 1) = \dfrac{\int_0^{1/4} f(1, x_2)\,dx_2}{f_1(1)}$

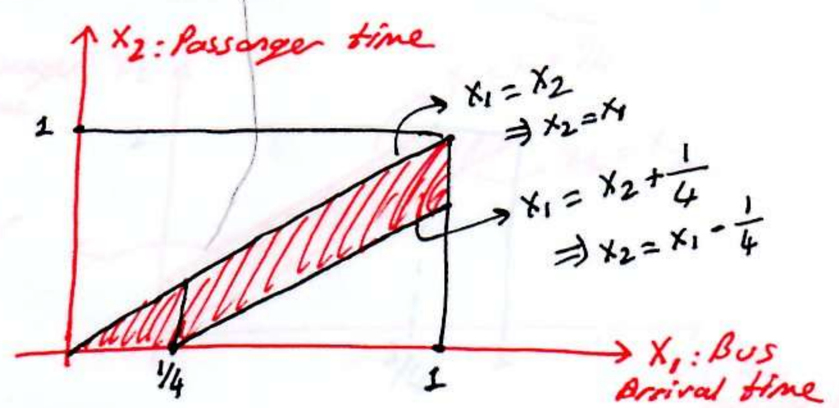
$= \dfrac{\int_0^{1/4} dx_2}{1/2} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} = 0{,}5$

5.12 A bus arrives at a bus stop at a randomly selected time within a 1-hour period. A passenger arrives at the bus stop at a randomly selected time within the same hour. The passenger will wait for the bus up to one-quarter of an hour. What is the probability that the passenger will catch the bus? [*Hint*: Let X_1 denote the bus arrival time and X_2 the passenger arrival time. If these arrivals are independent, then

$$f(x_1, x_2) = \begin{cases} 1 & 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Now find $P(X_2 \leq X_1 \leq X_2 + 1/4)$.]

5.12)



Note that, bus does NOT wait the passenger. If we had the problem "Two friends will meet and the one who came before will wait for one-quarter of hour" Then, the probability would be $P(X_2 - 1/4 \leq X_1 \leq X_2 + \frac{1}{4})$

$$P(X_2 \leq X_1 \leq X_2 + 1/4) = \int_0^{1/4}\int_0^{x_1} f(x_1,x_2)\,dx_2\,dx_1 + \int_{1/4}^{1}\int_{x_1-1/4}^{x_1} f(x_1,x_2)\,dx_2\,dx_1$$

$$= \int_0^{1/4}\underbrace{\int_0^{x_1} 1\,dx_2}_{x_1}\,dx_1 + \int_{1/4}^{1}\underbrace{\int_{x_1-1/4}^{x_1} 1.\,dx_2}_{x_1-(x_1-\frac{1}{4})=\frac{1}{4}}\,dx_1 = \int_0^{1/4} x_1\,dx_1 + \int_{1/4}^{1}\frac{1}{4}\,dx_1$$

$$= \frac{x_1^2}{2}\Big]_0^{1/4} + \frac{1}{4}x_1\Big]_{1/4}^{1} = \frac{1}{32} + \frac{1}{4}\left(1-\frac{1}{4}\right) = \frac{1}{32} + \frac{3}{16} = \frac{7}{32}$$

5.21 A particular fast-food outlet is interested in the joint behavior of the random variables Y_1, the total time between a customer's arrival at the store and his leaving the service window, and Y_2, the time that the customer waits in line before reaching the service window. Since Y_1 contains the time a customer waits in line, we must have $Y_1 \geq Y_2$. The relative frequency distribution of observed values of Y_1 and Y_2 can be modeled by the probability density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1} & 0 \leq y_2 \leq y_1 < \infty \\ 0 & \text{elsewhere} \end{cases}$$

- **a** Find $P(Y_1 < 2, Y_2 > 1)$.
- **b** Find $P(Y_1 \geq 2Y_2)$.
- **c** Find $P(Y_1 - Y_2 \geq 1)$. [*Note*: $Y_1 - Y_2$ denotes the time spent at the service window.]
- **d** Find the marginal density functions for Y_1 and Y_2.

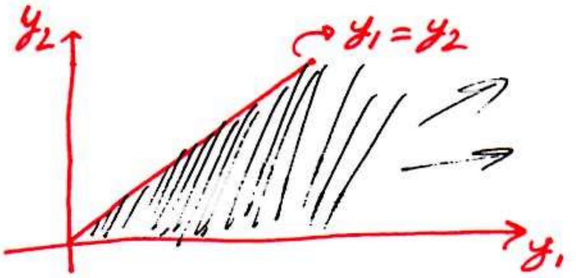
5.22 Refer to Exercise 5.21. If a customer's total waiting time plus service time is known to be more than 2 minutes, find the probability that the customer waited less than 1 minute to be served.

5.23 Refer to Exercise 5.21. The random variable $Y_1 - Y_2$ represents the time spent at the service window.

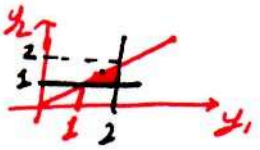
- **a** Find $E(Y_1 - Y_2)$.
- **b** Find $V(Y_1 - Y_2)$.
- **c** Is it highly likely that a customer would spend more than 2 minutes at the service window?

5.24 Refer to Exercise 5.21. Suppose a customer spends a length of time y_1 at the store. Find the probability that this customer spends less than half of that time at the service window.

5.21)



a) $P(Y_1 < 2,\ Y_2 > 1) = \int_1^2 \int_{y_2}^2 f(y_1, y_2)\, dy_1\, dy_2$

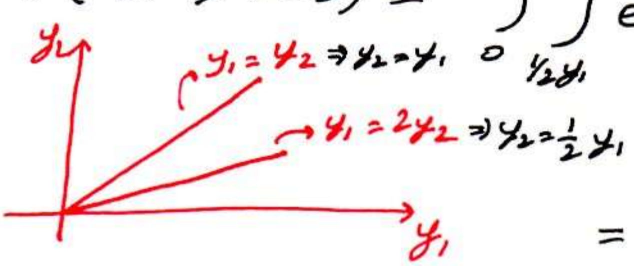


$$= \int_1^2 \int_{y_2}^2 e^{-y_1}\, dy_1\, dy_2 = \int_1^2 \left[-e^{-y_1}\right]_{y_1=y_2}^{y_1=2} dy_2 = \int_1^2 \left(e^{-y_2} - e^{-2}\right) dy_2$$

$$= \left(-e^{-y_2} - e^{-2} y_2\right)\Big|_1^2 = \left(e^{-y_2} + e^{-2} y_2\right)\Big|_2^1 = \left(e^{-1} + e^{-2}\right) - \left(e^{-2} + 2e^{-2}\right)$$

$$= e^{-1} - 2e^{-2} = 0{,}0972$$

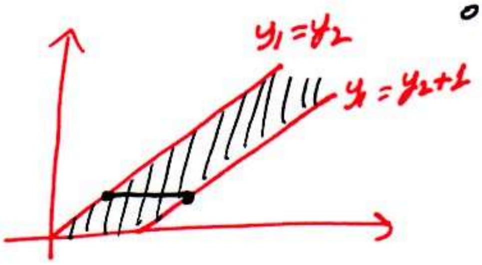
b) $P(Y_1 \geq 2Y_2) = \int_0^\infty \int_{\frac{1}{2}y_1}^{y_1} e^{-y_1}\, dy_2\, dy_1 = \int_0^\infty \left[y_2 \cdot e^{-y_1}\right]_{y_2 = \frac{1}{2}y_1}^{y_2 = y_1} dy_1$



$$= \int_0^\infty \left(y_1 - \tfrac{1}{2} y_1\right) e^{-y_1}\, dy_1$$

$$= \frac{1}{2} \int_0^\infty y_1 \cdot e^{-y_1}\, dy_1 = \frac{1}{2} \cdot \Gamma(2) = \frac{1}{2} \cdot 1! = \frac{1}{2}$$

c) $P(Y_1 - Y_2 < 1) = \int_0^\infty \int_{y_2}^{y_2+1} e^{-y_1}\, dy_1\, dy_2 = \int_0^\infty \left(-e^{-y_1}\right)\Big|_{y_1=y_2}^{y_1=y_2+1} dy_2$



$$= \int_0^\infty \left(-e^{-y_2-1} + e^{-y_2}\right) dy_2$$

$$= (1-e^{-1}) \underbrace{\int_0^\infty e^{-y_2}\, dy_2}_{\Gamma(1)=0!=1} = 1-e^{-1}$$

$$P(Y_1 - Y_2 \geq 1) = 1 - P(Y_1 - Y_2 < 1) = 1-(1-e^{-1}) = e^{-1}$$

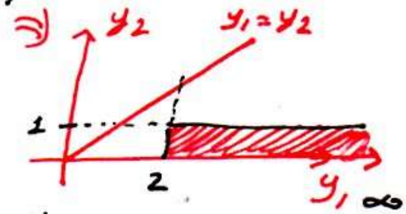
d) $f_1(y_1) = \int_0^{y_1} e^{-y_1}\, dy_2 = y_2 \cdot e^{-y_1}\Big]_{y_2=0}^{y_2=y_1} = y_1 \cdot e^{-y_1} \quad y_1 \geq 0$

$$f_2(y_2) = \int_{y_2}^\infty e^{-y_1}\, dy_1 = -e^{-y_1}\Big]_{y_1=y_2}^{y_1=\infty} = 0 - (-e^{-y_2}) = e^{-y_2} \quad y_2 \geq 0$$

5.22) Y_1 : Total time between arrival and leaving

Y_2 : Time customer wait in the queue

Then, $P(Y_2 < 1 \mid Y_1 > 2) = \dfrac{P(Y_2 < 1, Y_1 > 2)}{P(Y_1 > 2)}$



$$= \frac{\int_2^\infty \int_0^1 e^{-y_1}\, dy_2\, dy_1}{\underbrace{\int_2^\infty y_1 \cdot e^{-y_1}\, dy_1}} = \frac{\int_2^\infty \left[y_2 \cdot e^{-y_1}\right]_{y_2=0}^{y_2=1} dy_1}{\left[-e^{-y_1}(y_1+1)\right]_2^\infty} = \frac{\int_2^\infty e^{-y_1}\, dy_1}{1-\left(-e^{-y_1}(y_1+1)\right)_0^2} = \frac{-e^{y_1}\Big]_2^\infty}{1+3e^{-2}-1}$$

$u = y_1 \quad dv = e^{-y_1} dy_1$

$du = dy_1 \quad v = -e^{-y_1}$

$\int u\,dv = uv - \int v\,du = -y_1 \cdot e^{-y_1} - \int -e^{-y_1} dy_1 = -y_1 \cdot e^{-y_1} - e^{-y_1}$

$$= \frac{e^{-2}}{3e^{-2}} = \frac{1}{3}$$

5.23) $E(Y_1) = \int_0^\infty y_1 \cdot f_1(y_1)\, dy_1 = \int_0^\infty y_1 \cdot y_1 \cdot e^{-y_1}\, dy_1$

$= \int_0^\infty y_1^2 \cdot e^{-y_1}\, dy_1 = \Gamma(3) = 2! = 2$

$E(Y_1^2) = \int_0^\infty y_1^2 \cdot f_1(y_1)\, dy_1 = \int_0^\infty y_1^3 \cdot e^{-y_1}\, dy_1 = \Gamma(4) = 3! = 6$

$E(Y_2) = \int_0^\infty y_2 \cdot f_2(y_2)\, dy_2 = \int_0^\infty y_2 \cdot e^{-y_2}\, dy_2 = \Gamma(2) = 1! = 1$

$E(Y_2^2) = \int_0^\infty y_2^2 f_2(y_2)\, dy_2 = \int_0^\infty y_2^2 \cdot e^{-y_2}\, dy_2 = \Gamma(3) = 2! = 2$

$E(Y_1 \cdot Y_2) = \int_0^\infty \int_0^{y_1} y_1 \cdot y_2 \cdot f(y_1, y_2)\, dy_2\, dy_1$

$= \int_0^\infty \int_0^{y_1} y_1 \cdot y_2 \cdot e^{-y_1}\, dy_2\, dy_1 = \int_0^\infty y_1 \cdot e^{-y_1} \left(\int_0^{y_1} y_2\, dy_2 \right) dy_1$

$= \int_0^\infty y_1 \cdot e^{-y_1} \left[\frac{y_2^2}{2} \right]_{y_2=0}^{y_2=y_1} dy_1 = \int_0^\infty y_1 \cdot e^{-y_1} \cdot \frac{y_1^2}{2}\, dy_1 = \frac{1}{2} \int_0^\infty y_1^3 \cdot e^{-y_1}\, dy_1$

$= \frac{1}{2} \cdot \Gamma(4) = \frac{1}{2} \cdot 3! = 3$

Then;

$Var(Y_1) = E(Y_1^2) - E^2(Y_1) = 6 - 2^2 = 2$

$Var(Y_2) = E(Y_2^2) - E^2(Y_2) = 2 - 1^2 = 1$

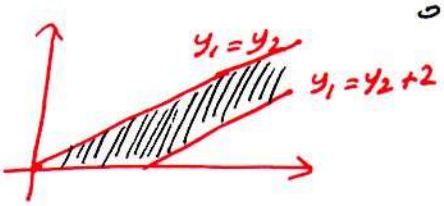
$Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) = 3 - 2 \cdot 1 = 1$

a) $E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = 2 - 1 = 1$

b) $Var(Y_1 - Y_2) = Var(Y_1) + Var(Y_2) - 2Cov(Y_1, Y_2)$

$= 2 + 1 - 2.1 = 1$

c) $P(Y_1 - Y_2 \leq 2) = \int_0^{\infty}\int_{y_2}^{y_2+2} e^{-y_1}\,dy_1\,dy_2 = \int_0^{\infty}\left(-e^{-y_1}\right)\Big|_{y_1=y_2}^{y_1=y_2+2} dy_2$



$= \int_0^{\infty}\left(-e^{-y_2-2} + e^{-y_2}\right) dy_2$

$= (1-e^{-2})\underbrace{\int_0^{\infty} e^{-y_2}\,dy_2}_{\Gamma(1)=0!=1} = 1 - e^{-2}$

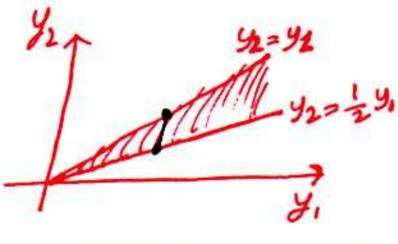
$P(Y_1 - Y_2 \geq 2) = 1 - P(Y_1 - Y_2 \leq 2) = 1 - (1-e^{-2}) = e^{-2}$

d) Find the correlation coefficient between Y_1 & Y_2.

Ans: $\rho = \dfrac{Cov(Y_1, Y_2)}{\sqrt{Var(Y_1).Var(Y_2)}} = \dfrac{1}{\sqrt{2.1}} = 0,707$

The linear relationship between Y_1 and Y_2 is moderate-to-strong $(0,5 < \rho < 1)$. This makes sense because Y_2 increases Y_1.

5.24) $P(Y_1 - Y_2 < 0,5Y_1) = P\left(\frac{1}{2}Y_1 - Y_2 \leq 0\right)$



$= \int_0^{\infty}\int_{\frac{1}{2}y_1}^{y_1} e^{-y_1}\,dy_2\,dy_1 = \int_0^{\infty}\left[y_2 . e^{-y_1}\right]_{y_2=\frac{1}{2}y_1}^{y_2=y_1} dy_1$

$= \int_0^{\infty}\left(y_1 - \frac{1}{2}y_1\right)e^{-y_1}\,dy_1 = \frac{1}{2}\int_0^{\infty} y_1 . e^{-y_1}\,dy_1 = \frac{1}{2}.\Gamma(2) = \frac{1}{2}$

Multinomial Distribution

Multinomial Distribution is extension of Binomial distribution. Remember, Binomial distribution has two outcomes for each trial, Success and Failure. For multinomial case, we have k outcomes. To illustrate, let there are 5 choices at a coffee machine, with probabilities $p_1, p_2, \ldots, p_5$ where $\sum p_i = 1$. If n people takes coffee from the machine, The probability distribution of y_1 from 1^{st} coffee, y_2 from 2^{nd} coffee, ..., y_5 from 5^{th} coffee is;

$$P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_5 = y_5) = \frac{n!}{y_1! \cdot y_2! \cdots y_5!} \cdot p_1^{y_1} \cdot p_2^{y_2} \cdots p_5^{y_5}$$

where $\sum_{i=1}^{5} y_i = n$ and $\sum_{i=1}^{5} p_i = 1$

Expectation and Variance for each Y_i is same as the Binomial Case; $E(Y_i) = n \cdot p_i$ and

$$Var(Y_i) = n \cdot p_i \cdot (1 - p_i)$$

Additionally, we have the Covariance Formula,

$$Cov(Y_i, Y_j) = -n \cdot p_i \cdot p_j \quad \text{for } i \neq j.$$

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5.26 The National Fire Incident Reporting Service reports that, among residential fires, approximately 73% are in family homes, 20% are in apartments, and the other 7% are in other types of dwellings. If four fires are independently reported in one day, find the probability that two are in family homes, one is in an apartment, and one is in another type of dwelling.

5.27 The typical cost of damages for a fire in a family home is \$20,000, the typical cost for an apartment fire is \$10,000, and the typical cost for a fire in other dwellings is only \$2,000. Using the information in Exercise 5.26, find the expected total damage cost for four independently reported fires.

5.26) $p_1 = 0{,}73,\ p_2 = 0{,}20$ and $p_3 = 0{,}07\ ;\ n = 4$

$$P(y_1 = 2, y_2 = 1, y_3 = 1) = \frac{4!}{2!.1!.1!} . 0{,}73^2 . 0{,}20^1 . 0{,}07^1 = 0{,}0895$$

5.27) $C = 20000 y_1 + 10000 y_2 + 2000 y_3$

$E(y_1) = np_1 = 0{,}73 \cdot 4 = 2{,}92\ ;\ E(y_2) = 0{,}20 . 4 = 0{,}8\ ;\ E(y_3) = 0{,}07 \cdot 4 = 0{,}28$

$E(C) = E(20000 y_1 + 10000 y_2 + 2000 y_3)$

$= 20000\, E(y_1) + 10000\, E(y_2) + 2000\, E(y_3)$

$= 20000 . 2{,}92 + 10000 . 0{,}8 + 2000 . 0{,}28 = 66960$

5.33 In a large lot of manufactured items, 10% contain exactly one defect and 5% contain more than one defect. If ten items are randomly selected from this lot for sale, the repair costs total

$$Y_1 + 3Y_2$$

where Y_1 denotes the number among the ten having one defect and Y_2 denotes the number with two or more defects. Find the expected value and variance of the repair costs. Find the variance of the repair costs.

5.33) $p_1 = 0{,}10\ ;\ p_2 = 0{,}05\ ;\ p_3 = 0{,}85\ ;\ n = 10$

$E(y_1) = 0{,}10 . 10 = 1\ ;\ E(y_2) = 0{,}05 . 10 = 0{,}5$

$Cov(y_1, y_2) = -10 . 0{,}10 . 0{,}05 = -0{,}05$

$Var(y_1) = 0{,}10 . 0{,}90 . 10 = 0{,}9\ ;\ Var(y_2) = 0{,}05 . 0{,}95 . 10 = 0{,}475$

$E(y_1 + 3y_2) = E(y_1) + 3E(y_2) = 1 + 3 . 0{,}5 = 2{,}5$

$Var(y_1 + 3y_2) = Var(y_1) + 9\,Var(y_2) + 6\,Cov(y_1, y_2) = 4{,}875$

5.34 Refer to Exercise 5.33. If Y denotes the number of items containing at least one defect among the ten sampled items, find the probability that

a Y is exactly 2.

b Y is at least 1.

Y is Binomial with $p = p_1 + p_2 = 0{,}10 + 0{,}05 = 0{,}15$

Then, $Y \sim \text{Binomial}(n=10;\ p=0{,}15)$

$$f(y) = \binom{10}{y} \cdot 0{,}15^{y} \cdot 0{,}85^{10-y}$$

a) $P(Y=2) = f(2) = \binom{10}{2} \cdot 0{,}15^{2} \cdot 0{,}85^{8} = 0{,}2759$

b) $P(Y \geq 1) = 1 - P(Y=0) = 1 - f(0) = 1 - \binom{10}{0} \cdot 0{,}15^{0} \cdot 0{,}85^{10}$

$= 0{,}8031$

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