Exponential Distribution.

$X \sim \text{Exponential}(\theta)$

$$f(x) = \begin{cases} \frac{1}{\theta} \cdot e^{-x/\theta} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = 1 - e^{-x/\theta} \qquad x > 0$$

$$\mu = E(X) = \theta \qquad \sigma^2 = Var(X) = \theta^2$$

* If events occur by Poisson Process, then interarrival times between events have exponential distribution.

$$N \sim \text{Poisson}(\theta) \iff T \sim \text{Exponential}\Big(\underbrace{\tfrac{1}{\theta}}_{\text{mean.}}\Big)$$

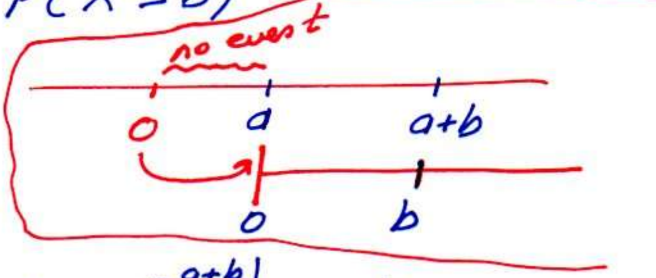
where N: # of events/unit time

T: Time between events (or time of next event)

If $\theta = 3$ events occur per hour, mean of time between events will be $\frac{1}{3} = 20$ minutes on the average.

* $P(X > a+b \mid X > a) = P(X > b)$

this is called memoryless.



$$P(X > a+b \mid X > a) = \frac{P(X > a+b,\ X > a)}{P(X > a)}$$

$$= \frac{P(X > a+b)}{P(X > a)} = \frac{1 - F(a+b)}{1 - F(a)} = \frac{1 - (1 - e^{-\frac{a+b}{\theta}})}{1 - (1 - e^{-\frac{a}{\theta}})} = e^{-\frac{b}{\theta}} = P(X > b)$$

4.31 The magnitudes of earthquakes recorded in a region of North America can be modeled by an exponential distribution with mean 2.4 as measured on the Richter scale. Find the probability that the next earthquake to strike this region will

- **a** exceed 3.0 on the Richter scale.
- **b** fall between 2.0 and 3.0 on the Richter scale.

4.32 Refer to Exercise 4.31. Of the next ten earthquakes to strike this region, find the probability that at least one will exceed 5.0 on the Richter scale.

4.31) $X \sim \text{Exponential}(\theta = 2{,}4)$

$$F(x) = 1 - e^{-x/2{,}4}$$

a) $P(X > 3{,}0) = 1 - P(X \leq 3{,}0) = 1 - F(3{,}0) = e^{-\frac{3{,}0}{2{,}4}} = 0{,}287$

b) $P(2{,}0 < X < 3{,}0) = F(3) - F(2) = e^{-\frac{2}{2{,}4}} - e^{-\frac{3}{2{,}4}} = 0{,}148$

4,32) $p = P(X > 5) = e^{-\frac{5}{2{,}4}} = 0{,}125$

y: # of earthquakes that pass 5,0

$y \sim \text{Binomial}(n = 10;\ p = 0{,}125)$

$$f(y) = \binom{10}{y} \cdot 0{,}125^{y} \cdot 0{,}875^{10-y}$$

$P(y \geq 1) = 1 - P(y = 0) = 1 - 0{,}875^{10} = 0{,}735$

4.43 In deciding how many customer service representatives to hire and in planning their schedules, it is important for a firm marketing electronic typewriters to study repair times for the machines. Such a study revealed that repair times have approximately an exponential distribution with a mean of 22 minutes.

- **a** Find the probability that a repair time will last less than 10 minutes.
- **b** The charge for typewriter repairs is \$50 for each half hour or part thereof. What is the probability that a repair job will result in a charge of \$100?
- **c** In planning schedules, how much time should be allowed for each repair so that the chance of any one repair time exceeding this allowed time is only 0.10?

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

4.43) $X \sim \text{Exponential}(\theta = 22)$

$$F(x) = 1 - e^{-\frac{x}{22}}$$

a) $P(X < 10) = F(10) = 1 - e^{-\frac{10}{22}} = 0,3653$

b) $P(\text{Charge of } 100) = P(60 < X < 90)$

$$= F(90) - F(60) = e^{-\frac{60}{22}} - e^{-\frac{90}{22}} = 0,0487$$

c) $P(X > t) = 0,10$

$1 - F(t) = 0,10$

$e^{-\frac{t}{22}} = 0,10$

$-\frac{t}{22} = \ln 0,10$

$t = -22 \ln 0,10 = 50,66$ min.

4.39 The breakdowns of an industrial robot follow a Poisson distribution with an average of 0.5 breakdown per 8-hour workday. The robot is placed in service at the beginning of the day.

a Find the probability that it will not break down during the day.

b Find the probability that it will work for at least 4 hours without breaking down.

c Does what happened the day before have any effect on your answers above? Why?

4.39) $X \sim \text{Exponential}(4)$

Because 0,5 breakdown / 8 hour $\Rightarrow$ mean breakdown time = 4

$$F(x) = 1 - e^{-x/4}$$

a) $P(X > 8) = 1 - P(X \leq 8) = 1 - F(8) = e^{-8/4} = 0,1353$

b) $P(X > 4) = 1 - P(X \leq 4) = 1 - F(4) = e^{-4/4} = 0,3679$

c) No, because of memoriless property.

4.42 The service times at teller windows in a bank were found to follow an exponential distribution with a mean of 3.2 minutes. A customer arrives at a window at 4:00 P.M.

a Find the probability that he will still be there at 4:02 P.M.

b Find the probability that he will still be there at 4:04 P.M. given that he was there at 4:02 P.M.

4.42) $X \sim \text{Exponential}(\theta = 3{,}2)$

$$F(x) = 1 - e^{-x/3{,}2}$$

a) $P(X > 2) = 1 - e^{-2/3{,}2} = 0{,}4647$

b) $P(X > 4 \mid X > 2) = P(X > 2) = 0{,}4647$

↳ by memoriless

4.44 Explosive devices used in a mining operation cause nearly circular craters to form in a rocky surface. The radii of these craters are exponentially distributed with a mean of 10 feet. Find the mean and variance of the area covered by such a crater.

$$r \sim \text{Exponential}(\theta = 10)$$

$$A(r) = 2\pi r^2$$

$$E(r) = 10 \quad ; \quad Var(r) = 10^2 = 100$$

$$E(r^2) = Var(r) + E^2(r) = 100 + 10^2 = 200$$

$$E[A(r)] = E(2\pi r^2) = 2\pi E(r^2) = 2\pi \cdot 200 = 400\pi$$

Gamma Distribution.

$$X \sim \text{Gamma}(\alpha; \beta)$$

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\,\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu = E(X) = \alpha\beta \qquad \sigma^2 = Var(X) = \alpha\beta^2$$

* when $\alpha = 1$, Gamma Distribution reduces to exponential distribution with mean β

$$\text{Gamma}(\alpha = 1, \beta) \equiv \text{Exponential}(\beta)$$

* Let $X_i \overset{i.i.d}{\sim} \text{Gamma}(\alpha; \beta) \quad i = 1, 2, \ldots, n$

Then, $Y = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n.\alpha; \beta)$

* So, if time until next event $T_i \sim \text{Exponential}(\beta)$
Then, time until n^{th} event: $Y = T_1 + T_2 + \cdots + T_n$

$$Y \sim \text{Gamma}(n; \beta)$$

(This is also called Erlang Distribution)

* $\Gamma(\alpha)$: Gamma function

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} \cdot e^{-x} \, dx \quad ; \Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1) \text{ and } \Gamma(1/2) = \sqrt{\pi}$$

when α is an integer, $\Gamma(\alpha) = (\alpha - 1)!$

* We do NOT have $F(x)$ of Gamma in explicit form. We sometimes use Tchebysheff's Thm. To find lower bounds for probabilities.

* Let $X_i \overset{i.i.d}{\sim} \text{Gamma}(\alpha_i; \beta)$

Then; $Y = \sum_{i=1}^{n} X_i \sim \text{Gamma}\left(\sum \alpha_i; \beta\right)$ which is a more general result of the corollary above.

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

4.46 Annual incomes for engineers in a certain industry have approximately a gamma distribution with $\alpha = 600$ and $\beta = 50$.

- **a** Find the mean and variance of these incomes.
- **b** Would you expect to find many engineers in this industry with an annual income exceeding \$35,000?

$$X \sim Gamma(\alpha = 600; \beta = 50)$$

a) $E(X) = \alpha \cdot \beta = 600 \cdot 50 = 30000$

$Var(X) = \alpha \cdot \beta^2 = 60 \cdot 50^2 = 150000$

b) $\mu = 30000, \ \sigma = \sqrt{150000} = 387,3$

$30000 + k \cdot 387,3 = 35000$

$k = \dfrac{35000 - 30000}{387,3} = 1,29$

$P(X > 35000) \leq \dfrac{1}{k^2} = \dfrac{1}{1,29^2} = 0,6$

At most 60% of the engineers can earn more than 35 000. ~~If~~ it many engineers? It depends :)

4.48 Customers arrive at a checkout counter according to a Poisson process with a rate of two per minute. Find the mean, variance, and probability density function of the waiting time between the opening of the counter and

- **a** the arrival of the second customer.
- **b** the arrival of the third customer.

4.48) $\dfrac{1}{\theta} = 2/\text{minute}, \quad \theta = \dfrac{1}{2}$ minutes.

a) $Y = T_1 + T_2 \sim Gamma(\alpha = 2; \beta = \frac{1}{2})$

$E(Y) = 2 \cdot \frac{1}{2} = 1 \ ; \ Var(Y) = 2 \cdot (\frac{1}{2})^2 = \frac{1}{2} = 0,5$

b) $Y = T_1 + T_2 + T_3 \sim Gamma(\alpha = 3; \beta = \frac{1}{2})$

$E(Y) = 3 \cdot \frac{1}{2} = 1,5 \ ; \ Var(Y) = 3 \cdot (\frac{1}{2})^2 = \frac{3}{4} = 0,75$

4.47 The weekly downtime Y (in hours) for a certain industrial machine has approximately a gamma distribution with $\alpha = 3$ and $\beta = 2$. The loss, in dollars, to the industrial operation as a result of this downtime is given by

$$L = 30Y + 2Y^2$$

a Find the expected value and variance of L.

b Find an interval that will contain L on ~~approximately~~ at least 89% of the weeks that the machine is in use.

4.47) $y \sim Gamma(\alpha = 3; \beta = 2)$

$$f(y) = \frac{1}{\underbrace{\Gamma(3)}_{=2!}\, 2^3} \cdot y^2 \cdot e^{-y/2} = \frac{1}{16} \cdot y^2 \cdot e^{-y/2}$$

$$L = 30y + 2y^2$$

a) $E(L) = E(30y + 2y^2) = 30E(y) + 2E(y^2)$

$E(y) = 3.2 = 6 \qquad Var(y) = 3.2^2 = 12$

$E(y^2) = Var(y) + E^2(y) = 12 + 6^2 = 48$

$E(L) = 30.6 + 2.48 = 276$

$Var(L) = E(L^2) - E^2(L)$

$E(L^2) = E[(30y + 2y^2)^2] = E(900y^2 + 120y^3 + 4y^4)$

$= 900\,E(y^2) + 120\,E(y^3) + 4\,E(y^4)$

$$E(y^3) = \int_0^\infty y^3 f(y) = \int_0^\infty y^3 \cdot \frac{1}{16} y^2 \cdot e^{-y/2} dy = \frac{1}{16} \cdot \int_0^\infty y^5 e^{-y/2} dy$$

$$= \frac{1}{16} \cdot \underbrace{\Gamma(6)}_{=5!} \cdot 2^6 \underbrace{\int_0^\infty \frac{1}{\Gamma(6) \cdot 2^6} \cdot y^5 \cdot e^{-y/2} dy}_{=1 \text{ because it is } \int_0^\infty f(x)dx \text{ where } X \sim Gamma(\alpha = 6; \beta = 2)} = \frac{1}{16} \cdot 5! \cdot 2^6 = 480$$

Likewise,

$$E(y^4) = \int_0^\infty y^4 f(y) = \frac{1}{16} \cdot \Gamma(7) \cdot 2^7 = 5760$$

$$So;\ E(L^2) = 900 \cdot 48 + 120 \cdot 480 + 4 \cdot 5760 = 123840$$

$$Var(L) = 123840 - 276^2 = 47664$$

b) $\mu = 276$; $\sigma = \sqrt{47664} = 218,3$

$$1 - \frac{1}{k^2} = 0,89 \Rightarrow k = 3$$

$$\mu \pm 3 \cdot \sigma$$

$$276 \pm 3 \cdot 218,3$$

$(-378,9 ; 930,9)$ but loss cannot be negative.

So, the interval is; $(0 ; 930,9)$

4.50 The total sustained load on the concrete footing of a planned building is the sum of the dead load plus the occupancy load. Suppose the dead load X_1 has a gamma distribution with $\alpha_1 = 50$ and $\beta_1 = 2$, while the occupancy load X_2 has a gamma distribution with $\alpha_2 = 20$ and $\beta_2 = 2$. (Units are in kips, or thousands of pounds.)

- **a** Find the mean, variance, and probability density function of the total sustained load on the footing.
- **b** Find a value for the sustained load that should be exceeded only with probability less than 1/16.

a) $X_1 \sim Gamma(\alpha_1 = 50 ; \beta = 2)$

$X_2 \sim Gamma(\alpha_2 = 20 ; \beta = 2)$

Then, $y = X_1 + X_2 \sim Gamma(\alpha = 70 ; \beta = 2)$

$E(y) = 70 \cdot 2 = 140 \qquad Var(y) = 70 \cdot 2^2 = 280$

b) $\frac{1}{k^2} = \frac{1}{16} \Rightarrow k = 4 \qquad \mu + 4\sigma = 140 + 4 \cdot \sqrt{280} = 206,93$

Normal Distribution

$$X \sim \text{Normal}(\mu ; \sigma^2)$$

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

$$E(X) = \mu \qquad\qquad Var(X) = \sigma^2$$

Z: Standard Normal Distribution

$$Z \sim \text{Normal}(\mu = 0 ; \sigma^2 = 1^2)$$

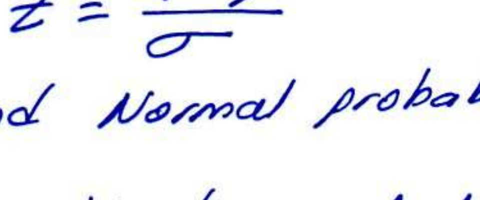
$$Z = \frac{X - \mu}{\sigma}$$

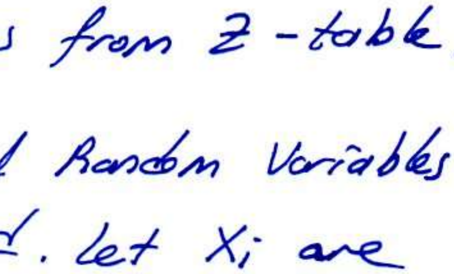
* We find Normal probabilities from Z-table.

* Linear combinations of Normal Random Variables are also normally distributed. Let X_i are ~~normal with~~ independent s.t. $X_i \sim \text{Normal}(\mu_i , \sigma_i^2)$

Then; $\quad Y = \sum_{i=1}^{n} X_i \sim \text{Normal}\left(\sum_{i=1}^{n} \mu_i ; \sum_{i=1}^{n} \sigma_i^2\right)$

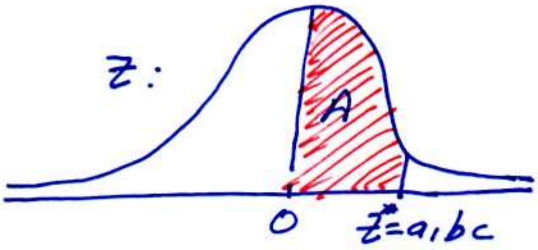
* Z - table

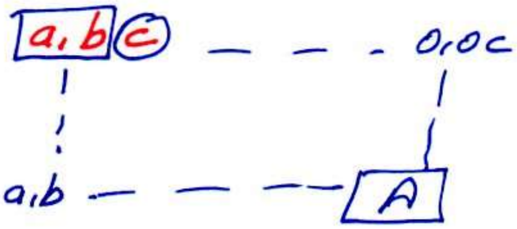




We have; (i) Total Area = 1
(ii) Half Area = 0,5
(iii) Symmetric areas are equal.

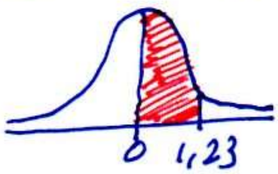
Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com





$A = P(0 < Z < a,bc)$

(I) $P(0 < z < 1{,}23) = ?$



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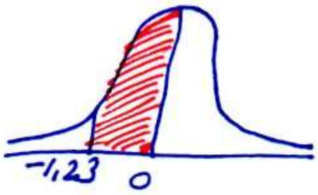
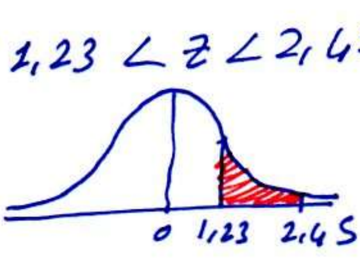


$P(0 < z < 1{,}23) = 0{,}3907 = P(-1{,}23 < z < 0)$

(II) $P(1{,}23 < z < 2{,}45) = ?$



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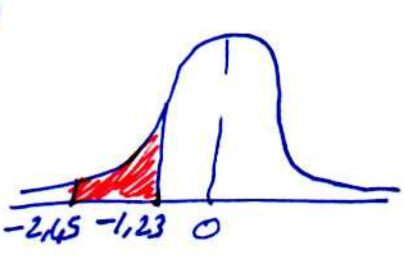
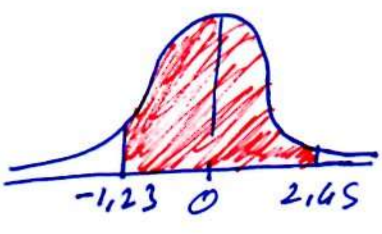


$P(1{,}23 < z < 2{,}45) = 0{,}4929 - 0{,}3907 = 0{,}1022$

$= P(-2{,}45 < z < -1{,}23)$

(III)



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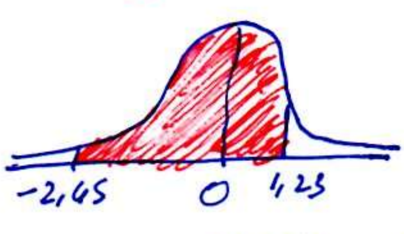
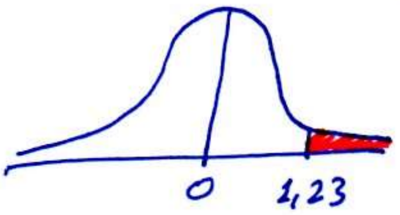
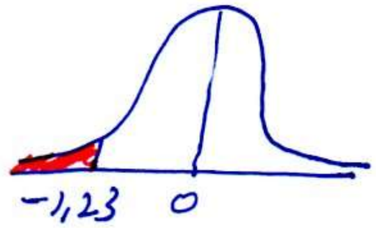


$P(-1{,}23 < z < 2{,}45) = 0{,}4929 + 0{,}3907 = 0{,}8836$

$= P(-2{,}45 < z < 1{,}23)$

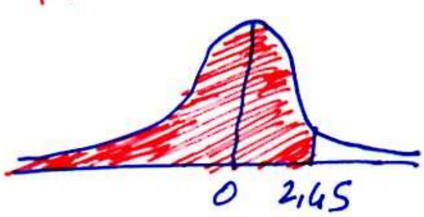
(IV) (i) $P(Z > 1,23) = ?$





$P(Z > 1,23) = 0,5 - 0,3907 = 0,1093 = P(Z < -1,23)$

(ii) $P(Z < 2,45) = ?$



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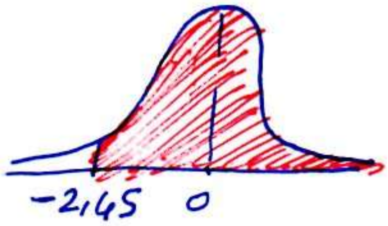
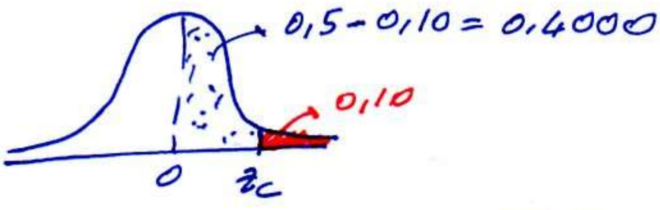


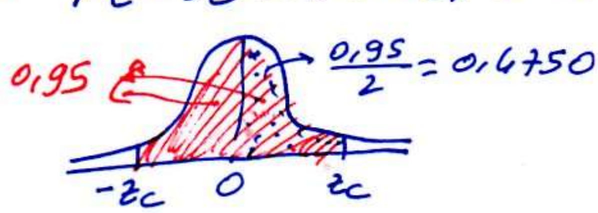
$P(Z < 2,45) = 0,5 + 0,4929 = 0,9929$
$= P(Z > -2,45)$

(V) (i) $P(Z > z_c) = 0,10 \Rightarrow z_c = ?$



$P(0 < Z < 1,28) = 0,3997 \Rightarrow z_c = 1,28$

(ii) $P(-z_c < Z < z_c) = 0,95 \Rightarrow z_c = ?$



$P(0 < Z < 1,96) = 0,4750 \Rightarrow z_c = 1,96$

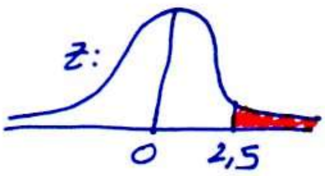
Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

4.57 The weekly amount spent for maintenance and repairs in a certain company has approximately a normal distribution with a mean of \$400 and a standard deviation of \$20. If \$450 is budgeted to cover repairs for next week, what is the probability that the actual costs will exceed the budgeted amount?

4.58 In the setting of Exercise 4.57, how much should be budgeted weekly for maintenance and repairs so that the budgeted amount will be exceeded with probability only 0.1?

4.57) $X \sim Normal(\mu = 400;\ \sigma^2 = 20^2)$

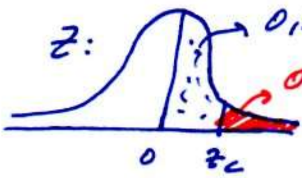
$$P(X > 450) = P\left(\frac{X-\mu}{\sigma} > \frac{450-400}{20}\right) = P(Z > 2{,}5)$$



$$= 0{,}5 - 0{,}4938$$

$$= 0{,}0062$$

4.58)



$0{,}5 - 0{,}10 = 0{,}4000$

Then; $z_c = 1{,}28$

$$z = \frac{X-\mu}{\sigma}$$

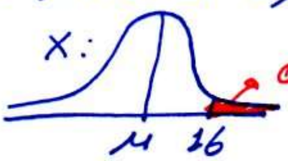
$$1{,}28 = \frac{X_c - 400}{20}$$

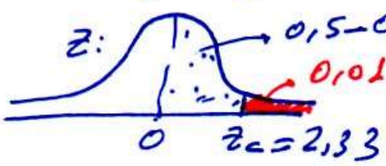
$$X_c = 400 + 1{,}28 \cdot 20 = 425{,}6$$

4.67 A machine for filling cereal boxes has a standard deviation of 1 ounce on ounces of fill per box. What setting of the mean ounces of fill per box will allow 16-ounce boxes to overflow only 1% of the time? Assume that the ounces of fill per box are normally distributed.

4.68 Refer to Exercise 4.67. Suppose the standard deviation σ is not known but can be fixed at certain levels by carefully adjusting the machine. What is the largest value of σ that will allow the actual value dispensed to be within 1 ounce of the mean with probability at least 0.95?

4.67) $X \sim Normal(\mu;\ \sigma^2 = 1^2)$





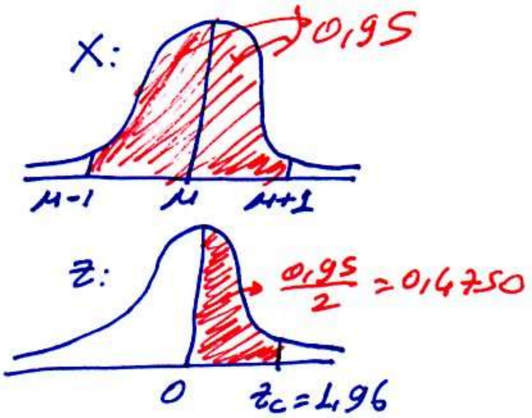
$0{,}5 - 0{,}01 = 0{,}4900$

$$Z = \frac{X-\mu}{\sigma}$$

$$2{,}33 = \frac{16-\mu}{1}$$

$$\mu = 16 - 2{,}33 = 13{,}67$$

4.68)



$$P(\underbrace{\mu-1 < X < \mu+1}) = 0,95$$

$$\frac{\mu-1-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{\mu+1-\mu}{\sigma}$$

$$P\left(-\frac{1}{\sigma} < Z < \frac{1}{\sigma}\right) = 0,95$$

$$\frac{1}{\sigma} = 1,96$$

$$\sigma = \frac{1}{1,96} = 0,51$$

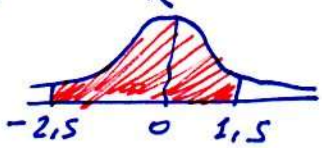
4.59 A machining operation produces steel shafts having diameters that are normally distributed with a mean of 1.005 inches and a standard deviation of 0.01 inch. Specifications call for diameters to fall within the interval 1.00 ± 0.02 inches. What percentage of the output of this operation will fail to meet specifications?

4.60 Refer to Exercise 4.59. What should be the mean diameter of the shafts produced to minimize the fraction not meeting specifications?

4.59) $X \sim Normal(\mu = 1,005;\ \sigma^2 = 0,01^2)$

$P(\text{Meet specifications}) = P(0,98 < X < 1,02)$

$$= P\left(\frac{0,98-1,005}{0,01} < Z < \frac{1,02-1,005}{0,01}\right) = P(-2,5 < Z < 1,5)$$



$= 0,4938 + 0,4332 = 0,927$

$P(\text{Fail to meet specifications}) = 1 - 0,927 = 0,073$

4.60) Mean diameter should be 1,00, the center specification.

Then: $P(\text{Meet Specifications}) = P(-2 < Z < 2) = 2 \cdot 0,4772 = 0,9544$

$P(\text{Fail to meet specifications}) = 1 - 0,9544 = 0,0456$

Beta Distribution

$X \sim Beta(\alpha; \beta)$

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha).\Gamma(\beta)} . x^{\alpha-1}(1-x)^{\beta-1} & 0<x<1 \\ 0 & o.w. \end{cases}$$

$$\mu = E(X) = \frac{\alpha}{\alpha+\beta} \qquad \sigma^2 = Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

4.73 Suppose X has a probability density function given by

$$f(x) = \begin{cases} kx^3(1-x)^2 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

a Find the value of k that makes this a probability density function.

b Find $E(X)$ and $V(X)$.

a) $X \sim Beta(\alpha = 4; \beta = 3)$

$$k = \frac{\Gamma(4+3)}{\Gamma(4)\,\Gamma(3)} = \frac{6!}{3!.2!} = 60$$

b) $E(X) = \dfrac{4}{3+4} = \dfrac{4}{7}$; $Var(X) = \dfrac{4.3}{(4+3)^2(4+3+1)} = 0{,}0306$

4.76 The percentage of impurities per batch in a certain type of industrial chemical is a random variable X having the probability density function

$$f(x) = \begin{cases} 12x^2(1-x) & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

a Suppose a batch with more than 40% impurities cannot be sold. What is the probability that a randomly selected batch will not be allowed to be sold?

b Suppose the dollar value of each batch is given by

$$V = 5 - 0.5X$$

Find the expected value and variance of V.

4.76) $X \sim Beta(\alpha=3;\ \beta=2)$

a) $P(X>0{,}40)=\int_{0{,}40}^{1} f(x)\,dx=\int_{0{,}40}^{1} 12x^2(1-x)\,dx$

$=12\int_{0{,}40}^{1}(x^2-x^3)\,dx=12\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_{0{,}40}^{1}=12\left[\left(\frac{1^3}{3}-\frac{1^4}{4}\right)-\left(\frac{0{,}4^3}{3}-\frac{0{,}4^4}{4}\right)\right]$

$=0{,}8208$

b) $E(X)=\frac{3}{3+2}=0{,}6$; $Var(X)=\frac{3.2}{(3+2)^2(3+2+1)}=0{,}04$

$V=5-0{,}5X$

$E(V)=E(5-0{,}5X)=5-0{,}5E(X)=5-0{,}5.0{,}6=4{,}7$

$Var(V)=Var(5-0{,}5X)=0{,}5^2.Var(X)=0{,}01$

4.80 The proportion of pure iron in certain ore samples has a beta distribution with $\alpha = 3$ and $\beta = 1$.

a Find the probability that one of these samples will have more than 50% pure iron.

b Find the probability that two out of three samples will have less than 30% pure iron.

4.80) $X \sim Beta(\alpha=3;\ \beta=1)$

$f(x)=\frac{\Gamma(3+1)}{\Gamma(3)\Gamma(1)}.x^{3-1}.(1-x)^{1-1}=\frac{3!}{2!.0!}.x^2=3x^2 \quad 0\le x\le 1$

a) $P(X>0{,}5)=\int_{0{,}5}^{1}3x^2\,dx=3\left.\frac{x^3}{3}\right]_{0{,}5}^{1}=1^3-0{,}5^3=0{,}875$

b) $p=P(X<0{,}3)=\int_{0}^{0{,}3}3x^2\,dx=\left.x^3\right]_{0}^{0{,}3}=0{,}3^3=0{,}027$

$Y\sim Binomial(n=3;\ p=0{,}027)$

$P(Y=2)=\binom{3}{2}0{,}027^2.(1-0{,}027)^{3-2}=0{,}00213$

Weibull Distribution

$$X \sim \text{Weibull}(\gamma ; \theta)$$

$$f(x) = \begin{cases} \frac{\gamma}{\theta} x^{\gamma-1} e^{-x^{\gamma}/\theta} \\ 0 \quad \text{o.w.} \end{cases}$$

$$F(x) = 1 - e^{-x^{\gamma}/\theta} \quad x > 0$$

$$\mu = E(X) = \theta^{1/\gamma} \cdot \Gamma\left(1 + \frac{1}{\gamma}\right)$$

$$\sigma^2 = var(X) = \theta^{2/\gamma} \cdot \left\{ \Gamma\left(1 + \frac{2}{\gamma}\right) - \left[\Gamma\left(1 + \frac{1}{\gamma}\right)\right]^2 \right\}$$

* Remember; $\Gamma(\alpha) = (\alpha-1)\cdot\Gamma(\alpha-1)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

4.87 Resistors being used in the construction of an aircraft guidance system have lifelengths that follow a Weibull distribution with $\gamma = 2$ and $\theta = 10$ (measurements in thousands of hours).

a Find the probability that a randomly selected resistor of this type has a lifelength that exceeds 5,000 hours.

b If three resistors of this type are operating independently, find the probability that exactly one of the three resistors burns out prior to 5,000 hours of use.

c Find the mean and variance of the lifelength of such a resistor.

4.87) $X \sim \text{Weibull}(\gamma = 2 ; \theta = 10)$

a) $P(X > 5) = 1 - P(X \leq 5) = 1 - F(5)$

$$F(x) = 1 - e^{-x^2/10}$$

$$P(X > 5) = 1 - \left[1 - e^{-5^2/10}\right] = e^{-2,5} = 0{,}082$$

b) $Y \sim \text{Binomial}(n = 3 ; p = 0{,}082)$

$$P(Y = 1) = \binom{3}{1} \cdot 0{,}082^1 \cdot (1 - 0{,}082)^2 = 0{,}2075$$

c) $E(X) = 10^{1/2} \cdot \Gamma\left(1+\frac{1}{2}\right) = \sqrt{10} \cdot \Gamma\left(\frac{3}{2}\right) = \sqrt{10} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{3\sqrt{10}\sqrt{\pi}}{2}$

$= 8{,}407$

$Var(x) = 10^{2/2} \cdot \left\{ \Gamma\left(1+\frac{2}{2}\right) - \left[\Gamma\left(1+\frac{1}{2}\right)\right]^2 \right\}$

$= 10 \cdot \left[1! - \left(\frac{1}{2}\sqrt{\pi}\right)^2\right] = 2{,}146$

4.89 Maximum wind-gust velocities in summer thunderstorms were found to follow a Weibull distribution with $\gamma = 2$ and $\theta = 400$ (measurements in feet per second). Engineers designing structures in the areas in which these thunderstorms are found are interested in finding a gust velocity that will be exceeded only with probability 0.01. Find such a value.

4.89) $X \sim \text{weibull}\ (\gamma = 2;\ \theta = 400)$

$F(x) = 1 - e^{-x^2/400}$

$P(X > x) = 1 - F(x) = e^{-x^2/400} = 0{,}01$

$-x^2/400 = \ln 0{,}01$

$x^2 = -400 \ln 0{,}01 = 1842{,}07$

$x = \sqrt{1842{,}07} = 42{,}92$

Reliability

* Reliability of a product is its probability of working for a specified period of time.

Then:

$R(t) = P(X > t) = 1 - F(t) \qquad t \geq 0$

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

* Failure Rate Function, $r(t)$ represents the probability of failure during the time interval $(t, t+dt)$

$$r(t) = \frac{f(t)}{1 - F(t)} \quad t > 0$$

Ex: For exponential distribution, we have

$$R(t) = 1 - F(t) = 1 - (1 - e^{-t/\theta}) = e^{-t/\theta}$$

$$r(t) = \frac{\frac{1}{\theta} \cdot e^{-t/\theta}}{e^{-t/\theta}} = \frac{1}{\theta}$$

Note that failure rate of exponential is NOT a function of t. This is because of memoriless property. Then, T has a constant failure rate.

For weibull distribution, we have

$$R(t) = 1 - F(t) = 1 - [1 - e^{-t^{\alpha}/\theta}] = e^{-t^{\alpha}/\theta}$$

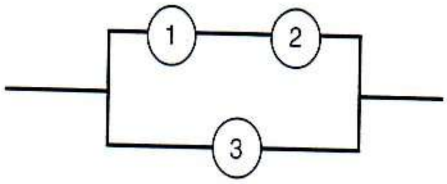
$$r(t) = \frac{\alpha/\theta \cdot t^{\alpha-1} e^{-t^{\alpha}/\theta}}{e^{-t^{\alpha}/\theta}} = \frac{\alpha}{\theta} \cdot e^{\alpha-1}$$

* Reliability of a system with serial components is given by $R_s(t) = [R(t)]^n$ and with parallel components is given by

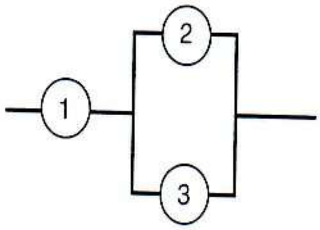
$$R_s(t) = 1 - [1 - R(t)]^n$$

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

4.93 For independently operating components with identical life distributions, find the system reliability for each of the following:



a



b

Which has the higher reliability?

4.93) System (a): $1 - R_S(t) = [1 - R^2(t)].[1 - R(t)]$

$$R_S(t) = 1 - [1 - R^2(t)].[1 - R(t)]$$

System (b): $R_S(t) = R(t).[1 - R(t)]^2$

Clearly, system a has higher reliability.

4.94 Suppose each relay in an electrical circuit has reliability 0.9 for a specified operating period of t hours. How could you configure a system of such relays to bring $R_s(t)$ up to 0.999?

We have $R(t) = 0,9$ and we want $R_S(t) = 0,999$.

By n circuits in series, we have;

$$1 - R_S(t) = [1 - R(t)]^n$$

$$1 - 0,999 = [1 - 0,9]^n$$

$$0,001 = 0,1^n$$

$n = 3$ components in series.

Moment generating Function

* Remember $E[g(x)] = \int g(x) f(x) dx$ where $f(x)$ is pdf of X.

Now, consider the function $g(t) = e^{tX}$

Its derivatives with respect to t are;

$$g^{(1)}(t) = \frac{dg(t)}{dt} = \frac{de^{tX}}{dt} = x \cdot e^{tX}$$

$$g^{(2)}(t) = \frac{d^2 g(t)}{dt^2} = \frac{d}{dt}\left[x \cdot e^{tX}\right] = x^2 \cdot e^{tX}$$

and in general, $g^{(k)}(t) = x^k \cdot e^{tX}$.

* The expectation $E(X^k)$ is called "k^{th} moment" of X.

Let $M^{(k)}(t) = E[g^{(k)}(t)] = E[X^k \cdot e^{tX}]$

Then, $M(t)$ is called "Moment generating function" because;

$$M^{(k)}(0) = E(X^k)$$

* Note that $\mu = E(X) = M^{(1)}(0)$ and

$$\sigma^2 = E(X^2) - \mu^2 = M^{(2)}(0) - [M^{(1)}(0)]^2$$

* mgf is unique. Namely, each pdf has a distinct mgf and the mapping is 1 to 1.

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

4.35 The length of time X to complete a certain key task in house construction is an exponentially distributed random variable with a mean of 10 hours. The cost C of completing this task is related to the square of the time to completion by the formula

$$C = 100 + 40X + 3X^2$$

a Find the expected value and variance of C.

b Would you expect C to exceed 2,000 very often?

4.35) a) The mgf of exponential distribution is;

$$m(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx} \cdot \frac{1}{\theta} \cdot e^{-x/\theta}\,dx$$

$$= \frac{1}{\theta} \cdot \int_0^{\infty} e^{-x(1/\theta - t)}\,dx = \frac{1}{\theta} \cdot \int_0^{\infty} e^{-x(1-\theta t)/\theta}\,dx$$

$$m(t) = \frac{1}{\theta} \cdot \frac{\theta}{1-\theta t} \underbrace{\int_0^{\infty} e^{-u}\,du}_{=\Gamma(1)=1}$$

$$\boxed{\frac{x(1-\theta t)}{\theta} = u \qquad \frac{(1-\theta t)}{\theta}dx = du \qquad dx = \frac{\theta}{1-\theta t}\,du}$$

$$m(t) = (1-\theta t)^{-1}$$

~~Then: $E(X) = m^{(1)}(0) =$~~

$$m^{(1)}(t) = -1.(-\theta).(1-\theta t)^{-1-1} = \theta.(1-\theta t)^{-2}$$

$$m^{(2)}(t) = -2.\theta.(-\theta).(1-\theta t)^{-3} = 2\theta^2.(1-\theta t)^{-3}$$

$$m^{(3)}(t) = 3!.\theta^3.(1-\theta t)^{-4} \text{ and } m^{(4)}(t) = 4!.(1-\theta t)^{-5}$$

So; $E(X) = \theta$; $E(X^2) = 2\theta^2$; $E(X^3) = 6\theta^3$ and $E(X^4) = 24\theta^4$

$$X \sim \text{Exponential}(\theta = 10)$$

$$C = 100 + 40X + 3X^2$$

$E(C) = E(100 + 40X + 3X^2) = 100 + 40E(X) + 3 \cdot E(X^2)$

$= 100 + 40 \cdot 10 + 3 \cdot 2 \cdot 10^2 = 1100$

$Var(C) = Var(100 + 40X + 3X^2) = Var(40X + 3X^2)$

$= E[(40X + 3X^2)^2] - E^2(40X + 3X^2)$

$E[(40X + 3X^2)^2] = E(1600X^2 + 240X^3 + 9X^4)$

$= 1600E(X^2) + 240E(X^3) + 9E(X^4)$

$= 1600 \cdot 2 \cdot 10^2 + 240 \cdot 6 \cdot 10^3 + 9 \cdot 24 \cdot 10^4 = 3920000$

$E^2[40X + 3X^2) = [40 \cdot 10 + 3 \cdot 2 \cdot 10^2]^2 = 1000000$

Then, $Var(C) = 3920000 - 1000000 = 2920000$

b) $2000 = 100 + 40X + 3X^2$

$X \approx 19{,}4$

then, $P(C > 2000) = P(X > 19{,}4) = e^{-1{,}94} = 0{,}1437$

Not very often.

4.96 Using the moment-generating function for the exponential distribution with mean θ, find $E(X^2)$. Use this result to show that $V(X) = \theta^2$.

4.96) we have shown that $E(X) = \theta$ and $E(X^2) = 2\theta^2$.

$Var(X) = E(X^2) - E^2(X) = 2\theta^2 - \theta^2 = \theta^2$

4.95 Show that a gamma distribution with parameters α and β has the moment-generating function

$$M(t) = (1-\beta t)^{-\alpha}$$

4.95) $X \sim Gamma(\alpha, \beta)$

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \cdot x^{\alpha-1} \cdot e^{-x/\beta} \quad x \geq 0$$

$$M(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \cdot \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \cdot x^{\alpha-1} \cdot e^{-x/\beta} dx$$

$$= \int_0^{\infty} \frac{1}{\Gamma(\alpha).\beta^{\alpha}} \cdot x^{\alpha-1} \cdot e^{-x\left(\frac{1}{\beta}-t\right)} dx$$

$$= \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \cdot x^{\alpha-1} \cdot e^{-x/\left(\frac{1}{\frac{1-\beta t}{\beta}}\right)} dx$$

$$\text{let } \frac{1}{\frac{1-\beta t}{\beta}} = \beta'$$

$$M(t) = \frac{1}{\beta^{\alpha}} \cdot (\beta')^{\alpha} \cdot \underbrace{\int_0^{\infty} \frac{1}{\Gamma(\alpha).(\beta')^{\alpha}} \cdot x^{\alpha-1} \cdot e^{-x/\beta'} dx}_{=1 \text{ because } \int_{-\infty}^{\infty} f(x)dx = 1}$$

$$M(t) = \frac{1}{\beta^{\alpha}} \cdot \frac{1}{\left(\frac{1-\beta t}{\beta}\right)^{\alpha}} = \frac{1}{\beta^{\alpha}} \cdot \frac{\beta^{\alpha}}{(1-\beta t)^{\alpha}} = \underline{(1-\beta t)^{-\alpha}}$$

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com