LECTURE NOTES
PROBABILITY | CHAPTER - 4

CONTINUOUS PROBABILITY DISTRIBUTIONS

* A random variable X is continuous if it takes values in an interval $(\alpha ; \beta)$. $f(x)$ is a "probability density function" if

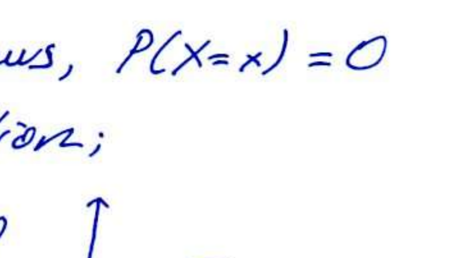
(i) $f(x) \geq 0 \quad \forall x$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

* Note that if x is continuous, $P(X=x) = 0$

We find probabilities by integration;

$$P(a < X < b) = \int_a^b f(x)dx = A$$



* The "(cummulative) distribution function" is;

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(k)dk$$

Then; $f(x) = \frac{d}{dx} F(x)$ and

$$P(a < X < b) = F(b) - F(a)$$

(Equality does not matter since $P(X=x)=0$)

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4.2 Suppose a random variable X has a probability density function given by

$$f(x) = \begin{cases} kx(1-x) & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

a Find the value of k that makes this a probability density function.

b Find $P(0.4 \le X \le 1)$.

c Find $P(X \le 0.4 | X \le 0.8)$.

d Find $F(b) = P(X \le b)$ and sketch the graph of this function.

4.2) a) $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 f(x)dx = \int_0^1 kx(1-x)dx = k\int_0^1 (x-x^2)dx$

$= k.\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = k\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{k}{6} = 1$

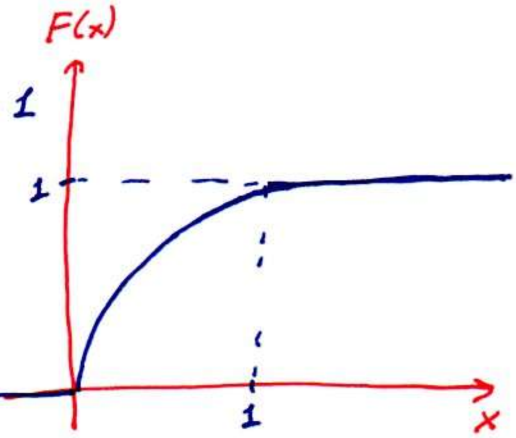
$\underline{k=6}$

$$f(x) = \begin{cases} 6x(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

d) $F(b) = \int_0^b f(x)dx = \int_0^b 6x(1-x)dx = 6\int_0^b (x-x^2)\,dx$

$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b = \cancel{6}.\left[\frac{3x^2-2x^3}{\cancel{6}}\right]_0^b = 3b^2 - 2b^3$

$$F(b) = \begin{cases} 0 & x < 0 \\ 3b^2 - 2b^3 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$



b) $P(0,4 \leq X \leq 1) = F(1) - F(0,4) = 1 - [3.0,4^2 - 2.0,4^3] = 0,648$

c) $P(X \leq 0,4 / X \leq 0,8) = \frac{P(X \leq 0,4,\ X \leq 0,8)}{P(X \leq 0,8)} = \frac{P(X \leq 0,4)}{P(X \leq 0,8)}$

$= \frac{F(0,4)}{F(0,8)} = \frac{3.0,4^2 - 2.0,4^3}{3.0,8^2 - 2.0,8^3} = 0,723$

4.4 An accounting firm that does not have its own computing facilities rents time from a consulting company. The firm must plan its computing budget carefully and hence has studied the weekly use of CPU time quite thoroughly. The weekly use of CPU time approximately follows the probability density function given below (measurements in hours):

$$f(x) = \begin{cases} \frac{3}{64}x^2(4-x) & 0 \leq x \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

a Find the distribution function $F(x)$ for weekly CPU time X.

b Find the probability that CPU time used by the firm will exceed 2 hours for a selected week.

c The current budget of the firm covers only 3 hours of CPU time per week. How often will the budgeted figure be exceeded?

d How much CPU time should be budgeted per week if this figure is to be exceeded with probability only 0.10?

4.4) a) $F(x) = \int_0^x f(k)dk = \int_0^x \frac{3}{64} k^2(4-k)\,dk = \frac{3}{64}\int_0^x [4k^2 - k^3]\,dk$

$= \frac{3}{64}\left(\frac{4k^3}{3} - \frac{k^4}{4}\right)\Big|_0^x = \frac{\cancel{3}}{64}\left[\frac{16k^3 - 3k^4}{\cancel{12}\,4}\right]_0^x = \frac{16x^3 - 3x^4}{256}$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{16x^3 - 3x^4}{256} & 0 \leq x \leq 4 \\ 1 & x > 4 \end{cases}$$

b) $P(X > 2) = 1 - P(X \leq 2) = 1 - F(2) = 1 - \frac{16.2^3 - 3.2^4}{256} = 0,6875$

c) $P(X>3) = 1 - P(X \leq 3) = 1 - F(3) = 1 - \frac{16 \cdot 3^3 - 3 \cdot 3^4}{256} = 0{,}2617$

d) $P(X > x) = F(x) = 0{,}10$

$$\frac{16x^3 - 3x^4}{256} = 0{,}10$$

$x \cong 3{,}429$ hours (by trying, no analytic solution)

4.8 The proportion of impurities X in certain copper ore samples is a random variable having probability density function

$$f(x) = \begin{cases} 12x^2(1-x) & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

If four such samples are independently selected, find the probability that

a exactly one has a proportion of impurities exceeding 0.5.

b at least one has a proportion of impurities exceeding 0.5.

4.8) $p = P(X > 0{,}5) = \int_{0{,}5}^{1} f(x)\,dx = \int_{0{,}5}^{1} 12x^2(1-x)\,dx$

$= 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_{0{,}5}^{1} = 12\left[\left(\frac{1^3}{3} - \frac{1^4}{4}\right) - \left(\frac{0{,}5^3}{3} - \frac{0{,}5^4}{4}\right)\right] = 0{,}6875$

$Y \sim \text{Binomial}(n = 4 ; p = 0{,}6875)$

$f(y) = \binom{4}{y} 0{,}6875^y \cdot 0{,}3125^{4-y}$

a) $P(Y=1) = \binom{4}{1} \cdot 0{,}6875^1 \cdot 0{,}3125^3 = 0{,}084$

b) $P(Y \geq 1) = 1 - P(Y=0) = 1 - 0{,}3125^4 = 0{,}9905$

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Expectation and Variance

* For a continuous random variable X, we replace Σ by $\int$. Remember, for discrete case we were summing $x f(x)$ ~~with~~ for expected value and $(x-\mu)^2 f(x)$ for variance.

$$\mu = E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$\sigma^2 = Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

OR; $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$; $\sigma^2 = E(X^2) - \mu^2$

* Also remember Tchebysheff's Theorem;

$$P(\,|X-\mu| < k\sigma\,) \geq 1 - \frac{1}{k^2}$$

OR $P(\mu - k.\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$

gives a lower bound for probability of X being in $k\sigma$ interval around μ.

* For linear functions of random variables, we have

$E(aX+b) = a.E(X) + b$ and

$Var(aX+b) = Var(aX) = a^2\, Var(X)$

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4.10 The proportion of time X that an industrial robot is in operation during a 40-hour work week is a random variable with probability density function

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

a Find $E(X)$ and $V(X)$.

b For the robot under study, the profit Y for a week is given by

$$Y = 200X - 60$$

Find $E(Y)$ and $V(Y)$.

c Find an interval in which the profit should lie for at least 75% of the weeks that the robot is in use. [*Hint*: Use Tchebysheff's Theorem.]

4.10) a) $E(X) = \int_0^1 x f(x) dx = \int_0^1 x \cdot 2x \, dx = 2 \cdot \frac{x^3}{3} \Big]_0^1 = 2 \cdot \frac{1}{3} = \frac{2}{3}$

$E(X^2) = \int_0^1 x^2 f(x) dx = \int_0^1 x^2 \cdot 2x \, dx = 2 \frac{x^4}{4} \Big]_0^1 = 2 \cdot \frac{1}{4} = \frac{1}{2}$

$Var(X) = E(X^2) - \mu^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{9}$

b) $E(Y) = E(200X - 60) = 200E(X) - 60 = 200 \cdot \frac{2}{3} - 60 = 73{,}33$

$Var(Y) = Var(200X - 60) = 200^2 \cdot Var(X) = 200^2 \cdot \frac{1}{9} = 4444{,}44$

c) $\mu = 73{,}3$; $\sigma = \sqrt{4444{,}44} = 66{,}67$

$1 - \frac{1}{k^2} = 0{,}75 \Rightarrow k = 2$

$\mu \pm 2 \cdot \sigma$ interval is;

$73{,}3 \pm 2 \cdot 66{,}67$

$(-60{,}04 ; 206{,}64)$

$(0 ; 206{,}64)$ since profit cannot be negative
(but maybe negative stands for loss)

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d) Would you expect the profit exceed \$300 very often? Why?

$$\mu + k.\sigma = 300$$

$$73,3 + k.66,67 = 300$$

$$k = \frac{300 - 73,3}{66,67} = 3,4$$

$$P(y<0 \text{ or } y>300) \leq \frac{1}{k^2} = \frac{1}{3,4^2} = 0,086$$

Only 8,6% of the time (at most) profit can be more than \$300. Not very often.

4.14 A retail grocer has a daily demand X for a certain food sold by the pound, such that X (measured in hundreds of pounds) has probability density function

$$f(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

(He cannot stock over 100 pounds.) The grocer wants to order $100k$ pounds of food on a certain day. He buys the food at 6¢ per pound and sells it at 10¢ per pound. What value of k will maximize his expected daily profit?

Let P is profit.

$$\text{if } X \leq K; \quad P = \overset{10-6}{4}X - 6(K - X) = 10X - 6K$$

$$\text{if } X > K; \quad P = 4K$$

$$\text{Then; } E(P) = \int_0^K (10X - 6K) f(x)dx + \int_K^1 4K f(x)dx$$

$$= \int_0^K (10x - 6K)3x^2 dx + \int_K^1 4K.3x^2 dx$$

$$= \int_0^K (30x^3 - 18x^2K)dx + \int_K^1 12Kx^2 dx$$

$$= \left[\frac{\cancel{30}^{\,15} x^4}{\cancel{4}_{\,2}} - \frac{\cancel{18}^{\,6} x^3}{\cancel{3}} K\right]_0^K + \left[\cancel{12}^{\,4} K \frac{x^3}{\cancel{3}}\right]_K^1$$

$$= \left(\frac{15K^4}{2} - 6K^4\right) + \left(4K - 4K^4\right) = 4K - \frac{5K^4}{2}$$

$$E(P) = 4K - \frac{5K^4}{2}$$

$$\frac{dE(P)}{dK} = 4 - \frac{20K^3}{2} = 4 - 10K^3 = 0$$

$$10K^3 = 4$$

$$K^3 = 0{,}4$$

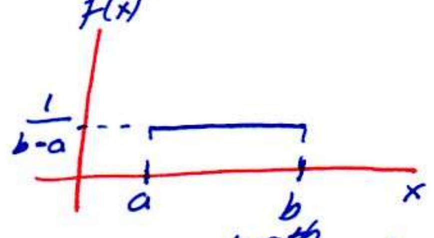
$K = 0{,}737$ will maximize $E(P)$

(Note that $\frac{d^2E(P)}{dK^2} = -30K^2 < 0$, so we have maximum)

Uniform Distribution.

$X \sim UNIFORM(a;b)$

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

* All equal length intervals have equal probability

$$\mu = E(X) = \frac{a+b}{2} \qquad \sigma^2 = Var(X) = \frac{(b-a)^2}{12}$$

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4.27 In tests of stopping distances for automobiles, those automobiles traveling at 30 miles per hour before the brakes are applied tend to travel distances that appear to be uniformly distributed between two points a and b. Find the probability that one of these automobiles

a stops closer to a than to b.

b stops so that the distance to a is more than three times the distance to b.

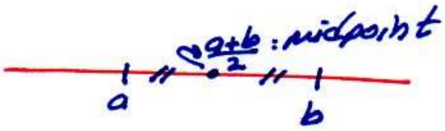
4.28 Suppose three automobiles are used in a test of the type discussed in Exercise 4.27. Find the probability that exactly one of the three travels past the midpoint between a and b.

4.27) $X \sim \text{Uniform}(a; b)$

$$F(x) = \frac{x-a}{b-a}$$

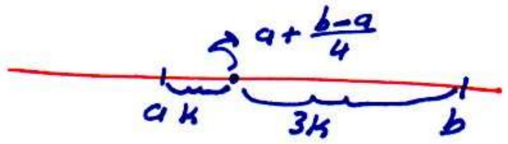
* Note that "at a random point between a and b" would also mean Uniform Distribution.

a)



$$P\left(X < \frac{a+b}{2}\right) = F\left(\frac{a+b}{2}\right) = \frac{\frac{a+b}{2} - a}{b-a} = \frac{1}{2}$$

b)



$$P\left(X < a + \frac{b-a}{4}\right) = F\left(a + \frac{b-a}{4}\right) = \frac{a + \frac{b-a}{4} - a}{b-a} = \frac{1}{4}$$

4.28) $p = P\left(X > \frac{a+b}{2}\right) = 1 - P\left(X \le \frac{a+b}{2}\right) = 1 - F\left(\frac{a+b}{2}\right) = \frac{1}{2}$

Y: # of cars that pass the midpoint

$Y \sim \text{Binomial}(n=3;\ p=\frac{1}{2})$

~~$P(Y \le 1) =$~~ $f(y) = \binom{3}{y}\left(\frac{1}{2}\right)^{y} \cdot \left(\frac{1}{2}\right)^{3-y} = \binom{3}{y}\left(\frac{1}{2}\right)^{3}$

$$P(Y=1) = \binom{3}{1}\left(\frac{1}{2}\right)^{3} = \frac{3}{8}$$

4.25 Arrivals of customers at a certain checkout counter follow a Poisson distribution. It is known that during a given 30-minute period one customer arrived at the counter. Find the probability that she arrived during the last 5 minutes of the 30-minute period.

4.25) N: Arrival of Customers

$N \sim Poisson(\lambda)$

By "Memoriless Property" of Poisson distribution, if an event is known to occur by Poisson Process, when it occurs will have a Uniform Distribution.

T: Time (ARRIVING) of the customer

$T \sim Uniform(0;30)$

$F(t) = \frac{t}{30}$

$P(T > 25) = 1 - P(T \leq 25) = 1 - F(25) = 1 - \frac{25}{30} = \frac{5}{30} = \frac{1}{6}$

4.24 According to Y. Zimmels (*AIChE Journal*, 29(4), 1983, pp. 669–676), the sizes of particles used in sedimentation experiments often have uniform distributions. It is important to study both the mean and variance of particle sizes, since in sedimentation with mixtures of various-size particles the larger particles hinder the movements of the smaller ones. Suppose spherical particles have diameters uniformly distributed between 0.01 and 0.05 centimeter. Find the mean and variance of the *volumes* of these particles. (Recall that the volume of a sphere is $\frac{4}{3}\pi r^3$.)

4.24) $2r \sim Uniform(0{,}01; 0{,}05)$

$r \sim Uniform(0{,}005; 0{,}025)$

$f(r) = \frac{1}{0{,}025 - 0{,}005} = 50 \quad 0{,}005 < r < 0{,}025$

$V(r) = \frac{4}{3}\pi r^3$

$E[V(r)] = \int_{0{,}005}^{0{,}025} V(r) f(r) dr = \int_{0{,}005}^{0{,}025} \frac{4}{3}\pi r^3 . 50 dr = \frac{\cancel{200}\ 50}{3}\pi . \frac{r^4}{\cancel{4}}\Big]_{0{,}005}^{0{,}025}$

$= \frac{50}{3}\pi (0{,}025^4 - 0{,}005^4) = 6{,}5 . 10^{-6}\pi$

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