DECISION ANALYSIS for BIM LECTURE NOTES | GAME THEORY LINEAR PROGRAMMING

GAME THEORY

Consider a game between two players. Each player have some strategies, known by both players. The game ends by choosing the strategies with a payoff value, specified by strategies.

For example, let two players will show one or two fingers at the same time. For obvious reasons that will be apparent soon, we'll call players "Row Player" and "Column Player". Let, if the fingers are the same, row player wins the sum (in TL) and otherwise, column player wins the sum.

A "game matrix" is the matrix in which strategies and payoffs are displayed. Note that, we'll consider payoffs in terms of row player. Namely, 4 means "row player wins 4 TL" and −3 means "Row player loses 3 TL = column player wins 3 TL"

Game Matrix:

	1	2
1	2	−3
2	−3	4

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14.1-3. Consider the following parlor game to be played between two players. Each player begins with three chips: one red, one white, and one blue. Each chip can be used only once.

To begin, each player selects one of her chips and places it on the table, concealed. Both players then uncover the chips and determine the payoff to the winning player. In particular, if both players play the same kind of chip, it is a draw; otherwise, the following table indicates the winner and how much she receives from the other player. Next, each player selects one of her two remaining chips and repeats the procedure, resulting in another payoff according to the following table. Finally, each player plays her one remaining chip, resulting in the third and final payoff.

Winning Chip	Payoff ($)
Red beats white	50
White beats blue	40
Blue beats red	30
Matching colors	0

Formulate this problem as a two-person, zero-sum game by identifying the form of the strategies and payoffs.

14.1.3) Row Player Strategies;
White; Red ; Blue

Colum Player Strategies;
White; Red ; Blue
(Note that they do Not need to be thesame)

	W	R	B
W	0	-50	40
R	50	0	-30
B	40	-30	0

* Remember, Given X is a random variable, the expected value of ~~X~~ ~~is~~ given by $E(X) = \sum x \cdot p_x$.
Then, given the probabilites of players to choose strategies, we can find expected Value (Gain) of Row Player, which is called "Value of the Game" if the strategies are optimal ones. (We'll define optimal strategy later)

Strategy		Player 2: 1	2	3
Player 1	1	4	2	−3
	2	−1	0	3
	3	2	3	−2

a) If column player always chooses strategy 1 and row player chooses 1 with prob. 0,3 and 2 with prob. 0,5, find expected gain of the row player.
b) Given the same probabilities for row player and column player chooses 1 with prob. 0.3 and 3 with prob. 0.7, find expected gain.

Answer: a)

		1
$p_1 = 0,3$	1	4
$p_2 = 0,5$	2	-1
$1-(0,3+0,5)$ $p_3 = 0,2$	3	2

$E(Gain) = 0,3 \cdot 4 + 0,5 \cdot (-1) + 0,2 \cdot 2$
$= 1,1$

b)

		1: $q_1 = 0,3$	3: $q_3 = 0,7$
1	$p_1 = 0,3$	4	-3
2	$p_2 = 0,5$	-1	3
3	$p_3 = 0,2$	2	-2

$E(Gain) = 0,3 \cdot 0,3 \cdot 4 + 0,3 \cdot 0,7 \cdot (-3) + 0,5 \cdot 0,3 \cdot (-1) + 0,5 \cdot 0,7 \cdot 3 + 0,2 \cdot 0,3 \cdot 2 + 0,2 \cdot 0,7 \cdot (-2) = 0,47$

Then, if probabilities of row player is known by column player, strategy (b) is better than strategy (a) for column player.

* However, strategies of players are NOT known by each other. Now, we'll learn how players "Eliminate" bad strategies.

1. Determine the saddle-point solution, the associated pure strategies, and the value of the game for each of the following games. The payoffs are for player A.

a)

	B_1	B_2	B_3	B_4
A_1	8	6	2	8
A_2	8	9	4	5
A_3	7	5	3	5

b)

	B_1	B_2	B_3	B_4
A_1	4	−4	−5	6
A_2	−3	−4	−9	−2
A_3	6	7	−8	−9
A_4	7	3	−9	5

Let's develop some way of "Logical Thinking" to play with a "good strategy". Consider the following rules;

(i) The row player will NOT choose a strategy i' if its payoffs are ≤ some other strategy i. Then, we say "strategy i dominates strategy i'" or equivalently "Strategy i' is eliminated by strategy i"

(ii) Likewise, Column player's strategy j' is eliminated by strategy j if every payoff in strategy j' is ≥ some other strategy j.

If we can reduce game matrix by elimination, the game is thought to be played within remaining strategies.

If only one strategy has left for both players, the game is called a strictly determined game and the resulting payoff is the saddle point solution, which is also value of the game.

a)

	B_1	B_2	B_3	B_4	
A_1	8	6	2	8	(8) El. by A_2
A_2	8	9	4	5	
A_3	7	5	3	5	(3) eliminated by A_2
	(5) el. by B_3	(6) el. by B_3		(7) el. by B_3	

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	B_1	B_2	B_3	B_4
A_1	8	6	2	9
A_2	8	9	4	5
A_3	7	5	3	9

Row player plays A_2; Column Player Plays B_3; strictly determined game, the saddle point is 4 (Row player wins 4TL)

b)

	B_1	B_2	B_3	B_4	
A_1	4	-4	-5	6	
A_2	~~-3~~	~~-4~~	~~-9~~	~~-2~~	by A_1
A_3	~~6~~	~~7~~	~~-8~~	~~-9~~	By A_1
A_4	~~7~~	~~3~~	~~-9~~	~~5~~	By A_1
	by B_3	by B_3		By B_3	

	B_1	B_2	B_3	B_4
A_1	4	-4	-5	6
A_2	-3	-4	-9	-2
A_3	6	7	-8	-9
A_4	7	3	-9	5

Row player Plays A_1; Column player Plays B_3; strictly determined game, the saddle point is -5 (Column player wins 5TL)

An alternative way to determine SADDLE POINT

We have an easier way of determining if the game is "strictly determined." However, we unfortunately can NOT eliminate rows or columns by this way.

(i) Find minimum value of each row: ○

(ii) Find max. value of each column: ▭

(iii) If they coincide, $\boxed{○}$ that's the saddle point and game is strictly determined. Otherwise, game is NOT strictly determined.

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Example: We may rework the previous example.

	B_1	B_2	B_3	B_4
A_1	$\boxed{8}$	6	2	$\boxed{9}$
A_2	$\boxed{8}$	$\boxed{9}$	$\boxed{4}$	5
A_3	7	5	3	$\boxed{9}$

Saddle Point: $4 = (A_2 ; B_3)$

	B_1	B_2	B_3	B_4
A_1	4	-4	-5	6
A_2	-3	-4	-9	-2
A_3	6	7	-8	-9
A_4	7	3	-9	5

Example: Note that, the game in Exercise 14.1-3 is NOT strictly determined.

	W	A	B
W	$\boxed{0}$	-50	-40
A	$\boxed{50}$	0	-30
B	$\boxed{40}$	-30	0

No coincidence $\boxed{0}$

As a final application, we'll learn how to find optimum probabilities for row and column player for a 2×2 game. Optimum probabilities will give an "(Expected) Value of the game" which (at the same time) is maximum for row player and minimum for colums player. The relevent formulas are given below:

	$q_1 = \frac{d-b}{T}$ I	$q_2 = \frac{a-c}{T}$ II	
$p_1 = \frac{d-c}{T}$ I	a	b	d-c
$p_2 = \frac{a-b}{T}$ II	c	d	a-b
	d-b	a-c	a+d-b-c=T

V: (Expected) Value of the game

$$V = \frac{ad-bc}{T}$$

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Example; Given the following game, find the optimum strategies and (Expected) Value of the game.

	B_1	B_2	B_3
A_1	3	6	1
A_2	5	2	3
A_3	4	2	-5

Answer;

(i) First, we make eliminations; (if any)

$B_1 \xrightarrow{4} B_2 \xrightarrow{6} B_3$, $B_1 \xrightarrow{5} B_3$

$A_1 \xrightarrow{1} A_2 \xrightarrow{3} A_3$, $A_1 \xrightarrow{2} A_3$

	~~B_1~~	B_2	B_3	
A_1	~~3~~	6	1	
A_2	~~5~~	2	3	
~~A_3~~	~~4~~	~~2~~	~~-5~~	by A_2 (3)
	by B_3			

(ii) Then, we find optimum strategies

	$q_2 = \frac{2}{6}$	$q_3 = \frac{4}{6}$	
	B_2	B_3	
$p_1 = \frac{1}{6}$ A_1	6	1	$3-2=1$
$p_2 = \frac{5}{6}$ A_2	2	3	$6-1=5$
	$3-1=2$	$6-2=4$	6

$$V = \frac{6.3-1.2}{6} = \frac{16}{6}$$

14.2-3. Consider the game having the following payoff table.

Strategy		Player 2: 1	2	3	4
Player 1	1	2	−3	−1	1
	2	−1	1	−2	2
	3	−1	2	−1	3

Determine the optimal strategy for each player by successively eliminating dominated strategies. Give a list of the dominated strategies (and the corresponding dominating strategies) in the order in which you were able to eliminate them.

14.2-3)

	~~1~~ (by 3)	2	3	~~4~~ (by 2)	
1	~~2~~	-3	-1	~~1~~	
~~2~~	~~-1~~	~~1~~	~~-2~~	~~2~~	by 3
3	~~-1~~	2	-1	~~3~~	

	$q_2=0$	$q_3=1$	
	2	3	
$p_1 = \frac{3}{5}$ 1	-3	-1	$-1-(2)=3$
$p_2 = \frac{2}{5}$ 3	2	-1	$-3-(-1)=-2$
	$-1-(-1)=0$	$-3-(2)=-5$	-5

$$V = \frac{(-3)(-1)-(2)(-1)}{-5} = \frac{-5}{5} = -1$$

LINEAR PROGRAMMING

Linear equations & Graphs.

* To graph a linear equation $ax+by+c=0$ (or $y=ax+b$);

Find the intersection points: $x=0 \Rightarrow y=y_0 \quad (0;y_0)$
$y=0 \Rightarrow x=x_0 \quad (x_0;0)$

and connect the points obtained.

We'll equivalently use $(x_1;x_2)$ representing $(x;y)$

* To find the intersection points of to lines, we solve them simultaneously.

Example. Find the intersection points of the equations:

(1) $6x_1+4x_2=24$
(2) $x_1+2x_2=6$
(3) $-x_1+x_2=1$

and draw their graph.

Answer: (1) $6x_1+4x_2=24$
$x_1=0 \Rightarrow x_2=6$
$x_2=0 \Rightarrow x_1=4$

(2) $x_1+2x_2=6$
$x_1=0 \Rightarrow x_2=3$
$x_2=0 \Rightarrow x_1=6$

(3) $-x_1+x_2=1$
$x_1=0 \Rightarrow x_2=1$
$x_2=0 \Rightarrow x_1=-1$

(1) & (2)
$6x_1+4x_2=24$
$-2/\ x_1+2x_2=6$

$6x_1+4x_2=24$
$+\ -2x_1-4x_2=-12$

$4x_1=12$
$x_1=3 \Rightarrow 3+2x_2=6$
$2x_2=3$
$x_2=1{,}5$

(1) & (3)
$6x_1+4x_2=24$
$6/\ -x_1+x_2=1$

$6x_1+4x_2=24$
$+\ -6x_1+6x_2=6$

$10x_2=30$
$x_2=3 \Rightarrow -x_1+3=1$
$x_1=2$

(2) & (3)
$x_1+2x_2=6$
$+\ -x_1+x_2=1$

$3x_2=7$
$x_2=2{,}33 \Rightarrow -x_1+2{,}33=1$
$x_1=1{,}33$

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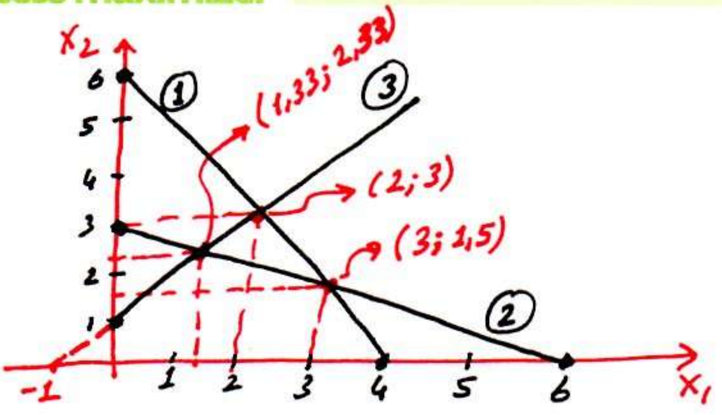


Linear inequalities & Feasible Region

* $ax+by+c \geq 0$ OR $ax+by+c \leq 0$. are linear inequalities we'll consider. An inequality divides the x-y plane into two regions. The region we'll "shade" (that satisfies the inequality) is; (i) Contain (0;0) for $ax+by+c \geq 0$

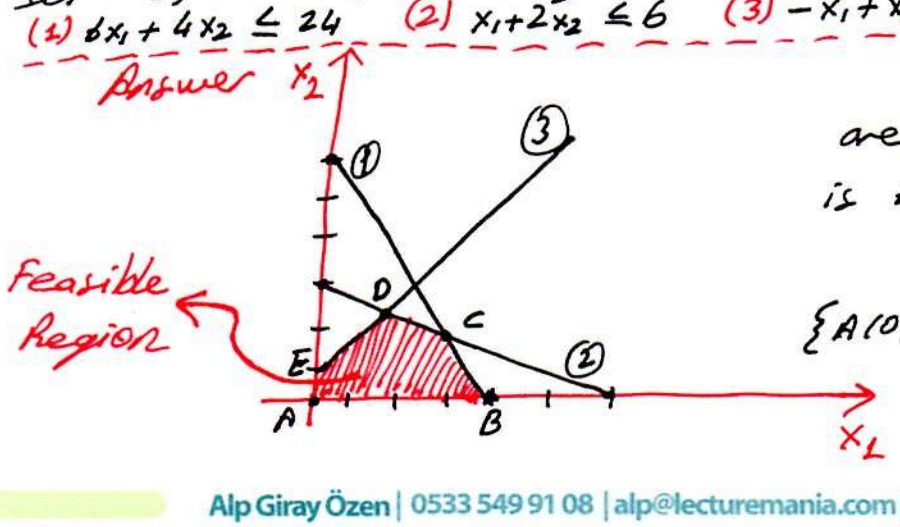
(ii) Do NOT Contain (0;0) for $ax+by+c \leq 0$

* The region that is satisfied by ALL inequalities is called "feasible region"

* The set of "corner points" of a feasible region is called "feasible set"

Example. Find the feasible region and the feasible set of the preceeding example with
(1) $6x_1+4x_2 \leq 24$ (2) $x_1+2x_2 \leq 6$ (3) $-x_1+x_2 \leq 1$

Answer



Since all inequalities are $\leq$, the feasible region is the most-inner region.

Feasible set is;

$\{A(0;0),\ B(4;0),\ C(3;1.5),\ D(1,33;2,33),\ E(0;1)\}$

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D 3.1-1.* For each of the following constraints, draw a separate graph to show the nonnegative solutions that satisfy this constraint.

(a) $x_1 + 3x_2 \le 6$

(b) $4x_1 + 3x_2 \le 12$

(c) $4x_1 + x_2 \le 8$

(d) Now combine these constraints into a single graph to show the feasible region for the entire set of functional constraints plus nonnegativity constraints.

Also find the feasible set

3.1.1) (i) Draw

(1) $x_1 + 3x_2 = 6$

(2) $4x_1 + 3x_2 = 12$

(3) $4x_1 + x_2 = 8$

(ii) Find intersection points

(iii) Shade the "innermost" region: Feasible Region and form the set of cornerpoints: Feasible Set

Then; we have,

(i) (1) $x_1 + 3x_2 = 6$
$x_1 = 0 \Rightarrow x_2 = 2$
$x_2 = 0 \Rightarrow x_1 = 6$

(2) $4x_1 + 3x_2 = 12$
$x_1 = 0 \Rightarrow x_2 = 4$
$x_2 = 0 \Rightarrow x_1 = 3$

(3) $4x_1 + x_2 = 8$
$x_1 = 0 \Rightarrow x_2 = 8$
$x_2 = 0 \Rightarrow x_1 = 2$

(ii) (1) & (2): $x_1 + 3x_2 = 6$
$-1/\ 4x_1 + 3x_2 = 12$

$x_1 + 3x_2 = 6$
$+\ -4x_1 - 3x_2 = -12$

$-3x_1 = -6 \Rightarrow x_1 = 2$
$2 + 3x_2 = 6 \Rightarrow x_2 = 1{,}33$

(1)&(3): $x_1 + 3x_2 = 6$
$-3/\ 4x_1 + x_2 = 8$

$x_1 + 3x_2 = 6$
$+\ -12x_1 - 3x_2 = -24$

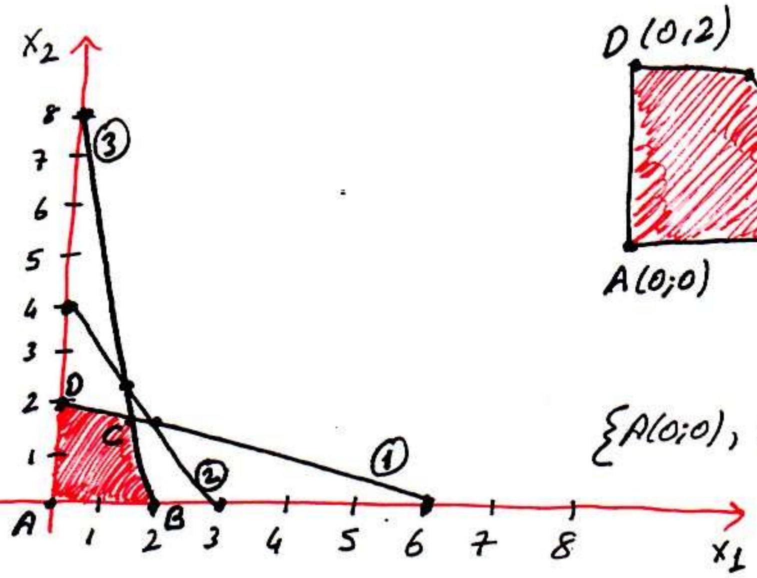
$-11x_1 = -18 \Rightarrow x_1 = 1{,}64$
$1{,}64 + 3x_2 = 6 \Rightarrow x_2 = 1{,}45$

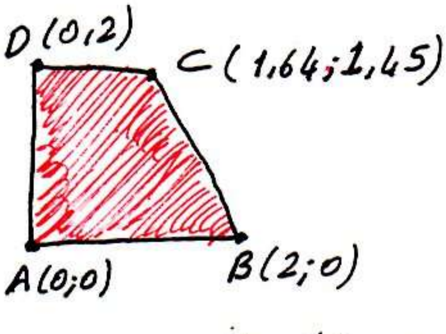
(2) & (3): $4x_1 + 3x_2 = 12$
$-1/\ 4x_1 + x_2 = 8$

$4x_1 + 3x_2 = 12$
$+\ -4x_1 - x_2 = -8$

$2x_2 = 4 \Rightarrow x_2 = 2$
$4x_1 + 2 = 8 \Rightarrow x_1 = 1{,}5$

(iii)





Feasible set is;

$\{A(0;0),\ B(2;0),\ C(1{,}64;\ 1{,}45),\ D(0,2)\}$

A linear programming question.

* The elements of a linear programming (LP) question are;

(i) Decision Variables; $x_1, x_2, \dots, x_n$

(ii) Objective function; $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

(iii) Constraint set; Some linear inequalities and $x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$

* Decision Variables (in general) represent amount/number of products 1, 2, ..., n or raw materials. The objective is to find optimum (Either Maximum (profit) or Minimum (cost)) value of the objective function z; given the constraint set AND determine the values of decision variables $x_1, x_2, \dots, x_n$ which gives this optimum value. LP questions are interested by "Optimization" or "Operations Research" studies of Engineers. The term "Management Science" is also used.

7. The Burroughs Garment Company manufactures men shirts and women blouses for Walmark Discount Stores. Walmark will accept all the production supplied by Burroughs. The production process includes cutting, sewing, and packaging. Burroughs employs 25 workers in the cutting department, 35 in the sewing department, and 5 in the packaging department. The factory works one 8-hour shift, only 5 days a week. The following table gives the time requirements and profits per unit for the two garments:

	Minutes per unit			
Garment	Cutting	Sewing	Packaging	Unit profit($)
Shirts	20	70	12	2.50
Blouses	60	60	4	3.20

Formulate a linear programming model for this problem.

7) Let x_1: Number of shirts produced / week.
x_2 : Number of blouses produced / week.

Our objective is to maximize weekly profit, which is given by $z = 2{,}50x_1 + 3{,}20x_2$

The available cutting, sewing and packing minutes/week will be our constraints. (They constraint us to make infinite profit.) The usual way representing the table is;

	Cutting	Sewing	Packing	PROFIT
x_1: Shirts	20	70	12	2,50
x_2: Blouses	60	60	4	3,20
AVAILABLE	60000	84000	12000	

Note that, available minutes are found as follows;

Cutting : 25 workers . 8 hours . 5 days . 60 minutes = 60000

Likewise; Sewing : 35 . 8 . 5 . 60 = 84000; Packing : 5 . 8 . 5 . 60 = 12000

Then, the LP model is;

Maximize $z = 2{,}50x_1 + 3{,}20x_2$

subject to constraints

$$20x_1 + 60x_2 \leq 60000$$
$$70x_1 + 60x_2 \leq 84000$$
$$12x_1 + 4x_2 \leq 12000$$
$$x_1 \geq 0, \; x_2 \geq 0$$

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3.4-7. Ralph Edmund loves steaks and potatoes. Therefore, he has decided to go on a steady diet of only these two foods (plus some liquids and vitamin supplements) for all his meals. Ralph realizes that this isn't the healthiest diet, so he wants to make sure that he eats the right quantities of the two foods to satisfy some key nutritional requirements. He has obtained the following nutritional and cost information:

	Grams of Ingredient per Serving		
Ingredient	x_1 Steak	x_2 Potatoes	Daily Requirement (Grams)
Carbohydrates	5	15	≥ 50
Protein	20	5	≥ 40
Fat	15	2	≤ 60
Cost per serving	$4	$2	

Ralph wishes to determine the number of daily servings (may be fractional) of steak and potatoes that will meet these requirements at a minimum cost.

(a) Formulate a linear programming model for this problem.

D,I (b) Use the graphical method to solve this model.

3.4-7) Note that; we "maximize" profit and "minimize" cost.

x_1: Number (amount) of daily serving of Steak

x_2: Number (amount) of daily serving of Potatoes.

Daily requirements are also given in the table. Do NOT be confused, table is "transposed" (costs are at the bottom)

We have the LP model;

Minimize $Z = 4x_1 + 2x_2$

subject to constraints

$$5x_1 + 15x_2 \geq 50$$

$$20x_1 + 5x_2 \geq 40$$

$$15x_1 + 2x_2 \leq 60$$

We'll see the "graphical solution" of an LP model later (at another example)

Solution of an LP problem

There are two ways to solve an LP problem.

(I) Graphical Method (only when there are two variables: x_1, x_2)

(II) Simplex Method (we will NOT see in detail)

(I) Graphical Method

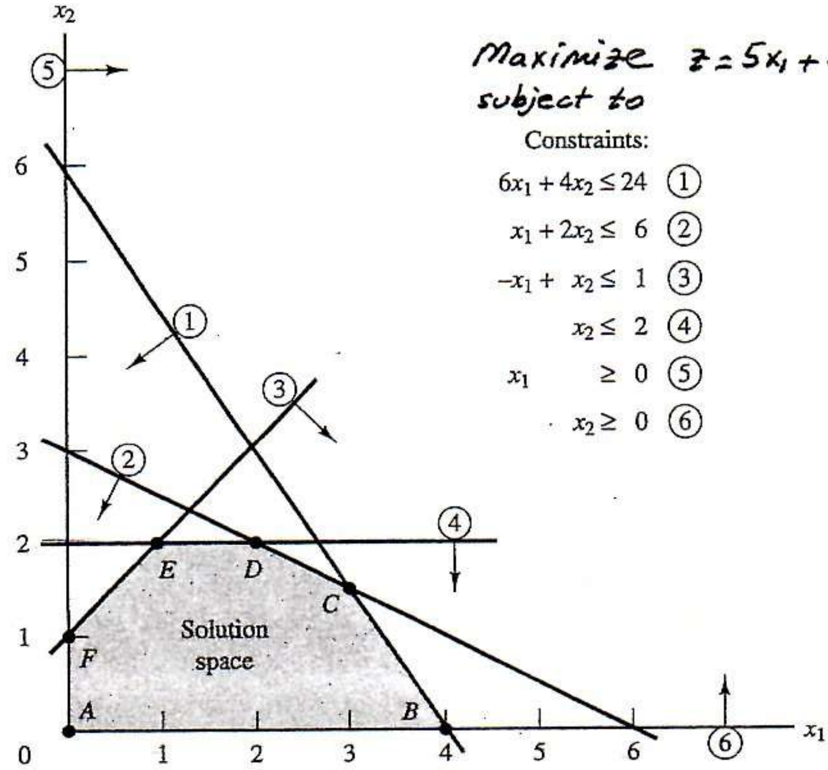
We have already seen how to obtain "feasible set" of constraints. Now, we'll use this set to find the optimum value of the objective function z. The procedure is; Just try each point in the feasible set for z and choose the maximum (minimum) value.

Example. let's return our first example by defining an objective function and adding a new constraint. Consider the following LP problem, whose feasible region and feasible set is also shown:

Maximize $z = 5x_1 + 4x_2$
subject to

Constraints:

$6x_1 + 4x_2 \leq 24$ (1)
$x_1 + 2x_2 \leq 6$ (2)
$-x_1 + x_2 \leq 1$ (3)
$x_2 \leq 2$ (4)
$x_1 \geq 0$ (5)
$x_2 \geq 0$ (6)



Feasible Point (x_1, x_2)	$z = 5x_1 + 4x_2$
$A(0;0)$	$z = 5.0 + 4.0 = 0$
$B(4;0)$	$z = 5.4 + 4.0 = 20$
$C(3;1,5)$	$z = 5.3 + 4.1,5 = 21$
$D(2;2)$	$z = 5.2 + 4.2 = 18$
$E(1;2)$	$z = 5.1 + 4.2 = 13$
$F(0;1)$	$z = 5.0 + 4.1 = 4$

Then, maximum z is 21 when $x_1 = 3$ and $x_2 = 1,5$

D,I **3.1-5.** Use the graphical method to solve the problem:

Maximize $Z = 10x_1 + 20x_2$,

subject to

$$-x_1 + 2x_2 \leq 15$$
$$x_1 + x_2 \leq 12$$
$$5x_1 + 3x_2 \leq 45$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

3.1-5) (1) $-x_1 + 2x_2 = 15$

$x_1 = 0 \Rightarrow x_2 = 7,5$

$x_1 = 5 \Rightarrow x_2 = 10$ → Not to go negative side; $x_2 = 0 \Rightarrow x_1 = -15$ is also possible

(2) $x_1 + x_2 = 12$

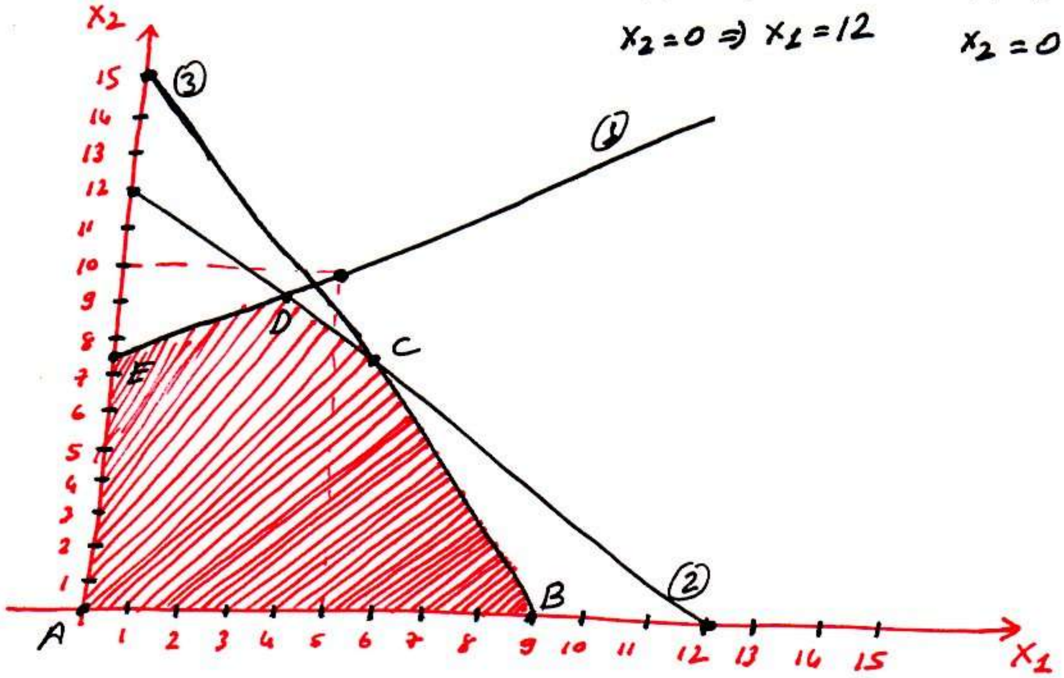
$x_1 = 0 \Rightarrow x_2 = 12$

$x_2 = 0 \Rightarrow x_1 = 12$

(3) $5x_1 + 3x_2 = 45$

$x_1 = 0 \Rightarrow x_2 = 15$

$x_2 = 0 \Rightarrow x_1 = 9$



C: (2) & (3): $x_1 + x_2 = 12 \quad (\cdot -3)$

$5x_1 + 3x_2 = 45$

$-3x_1 - 3x_2 = -36$

$+ \quad 5x_1 + 3x_2 = 45$

$2x_1 = 9 \Rightarrow x_1 = 4,5$

$4,5 + x_2 = 12 \Rightarrow x_2 = 7,5$

(Some error caused by drawing the graph)

D: (1) & (2): $-x_1 + 2x_2 = 15$

$+ \quad x_1 + x_2 = 12$

$3x_2 = 27$

$x_2 = 9$

$x_1 + 9 = 12 \Rightarrow x_1 = 3$

Feasible Point $(x_1; x_2)$	$Z = 10x_1 + 20x_2$
$A(0;0)$	$z = 10.0 + 20.0 = 0$
$B(9;0)$	$z = 10.9 + 20.0 = 90$
$C(4,5;7,5)$	$z = 10.4,5 + 20.7,5 = 195$
$D(3;9)$	$z = 10.3 + 20.9 = \underline{210}$
$E(0;7,5)$	$z = 10.0 + 20.7,5 = 150$

Then,
Maximum z is 210 when $x_1 = 3$ and $x_2 = 9$ (Point D)

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D,I **3.4-3.** Use the graphical method to solve this problem:

Minimize $Z = 15x_1 + 20x_2$,

subject to

$$x_1 + 2x_2 \geq 10$$
$$2x_1 - 3x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

3.4-3) (1) $x_1 + 2x_2 = 10$

$x_1 = 0 \Rightarrow x_2 = 5$

$x_2 = 0 \Rightarrow x_1 = 10$

(2) $2x_1 - 3x_2 = 6$

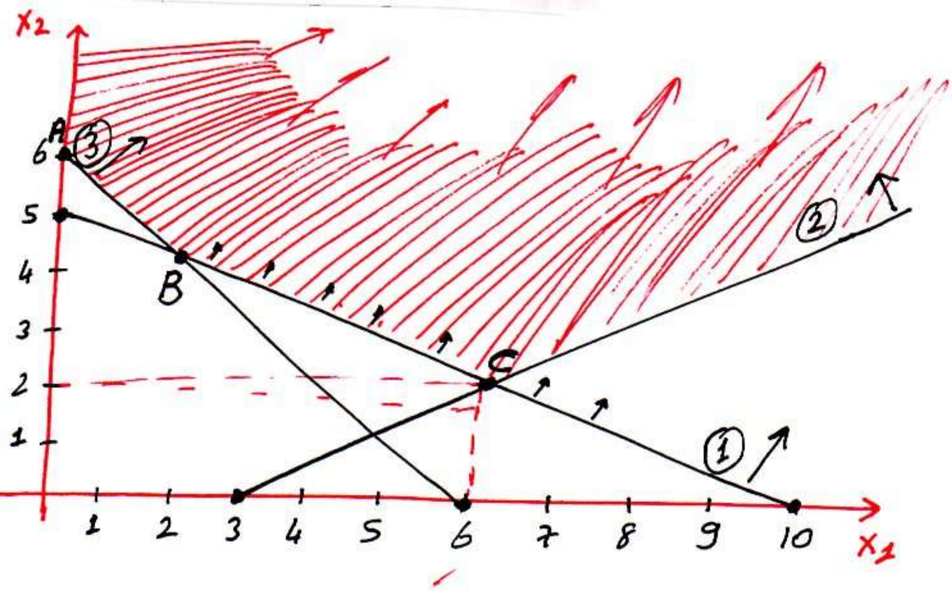
$x_1 = 6 \Rightarrow x_2 = 2$

$x_2 = 0 \Rightarrow x_1 = 3$

(3) $x_1 + x_2 = 6$

$x_1 = 0 \Rightarrow x_2 = 6$

$x_2 = 0 \Rightarrow x_1 = 6$



B: (1) & (3): $x_1 + 2x_2 = 10$

$-1/ \; x_1 + x_2 = 6$

$x_1 + 2x_2 = 10$

$+ \; -x_1 - x_2 = -6$

$x_2 = 4$

$x_1 + 4 = 6 \Rightarrow x_1 = 2$

C: (1) & (2): $3/ \; x_1 + 2x_2 = 10$

$2/ \; 2x_1 - 3x_2 = 6$

$3x_1 + 6x_2 = 30$

$+ \; 4x_1 - 6x_2 = 12$

$7x_1 = 42 \Rightarrow x_1 = 6$

$6 + 2x_2 = 10 \Rightarrow x_2 = 2$

Feasible point $(x_1; x_2)$	$Z = 15x_1 + 20x_2$
A(0;6)	$Z = 15.0 + 20.6 = 120$
B(2;4)	$Z = 15.2 + 20.4 = 110$
C(6;2)	$Z = 15.6 + 20.2 = 130$

Then,
Minimum Z is 110
when $x_1 = 2$ and $x_2 = 4$
(Point B)

(II) Simplex Method

We learned how to solve a two-variable LP problem by graphical method. A more general solution is obtained by simplex method. We'll only see a special case, maximization problem only with $\leq$ constrains. Also we won't see the solution in detail, we'll only learn how to form a simplex tableau and make the first iteration

4.4-6. Consider the following problem.

Maximize $\quad Z = 3x_1 + 5x_2 + 6x_3$,

subject to

$$2x_1 + x_2 + x_3 \leq 4$$
$$x_1 + 2x_2 + x_3 \leq 4$$
$$x_1 + x_2 + 2x_3 \leq 4$$
$$x_1 + x_2 + x_3 \leq 3$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

4.4.6) First, consider the $\leq$ inequalities. To form simplex table, we need linear equations "=". Then, we add a slack variable "s_i" to each constraint;

(1) $2x_1 + x_2 + x_3 + s_1 = 4$

(2) $x_1 + 2x_2 + x_3 + s_2 = 4$

(3) $x_1 + x_2 + 2x_3 + s_3 = 4$

(4) $x_1 + x_2 + x_3 + s_4 = 3$

We also make objective function equal to 0 and obtain;

$$z - 3x_1 - 5x_2 - 6x_3 = 0$$

Now, we need an initial solution, which we will construct the initial simplex tableau. Initially, we'll equate s_i's to the "right hand side" values, and all other variables to 0. Then, since there is no s_i in z, the objective function value is also 0. We have;

$s_1 = 4$; $s_2 = 4$; $s_3 = 4$; $s_4 = 3$ and

$x_1 = x_2 = x_3 = 0$ so;

$z = 3.0 - 5.0 - 6.0 = 0.$

The initial simplex tableau is; RHS (Right hand side)

Basic	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	b
(0) z	1	-3	-5	-6	0	0	0	0	0
(1) s_1	0	2	1	1	1	0	0	0	4
(2) s_2	0	1	2	1	0	1	0	0	4
(3) s_3	0	1	1	2	0	0	1	0	4
(4) s_4	0	1	1	1	0	0	0	1	3

To make an iteration, we will find "pivot element" by following steps;

(i) Find the "Most Negative Value" of Row 0.

(ii) Make a "Minimum Ratio Test": divide each element of the RHS constraint to "Most Negative Column"'s corresponding element, select Minimum Positive value's row as "Pivot Row"

(iii) The intersection of "Most Negative Value Column" and "Pivot Row" is "Pivot element". Divide all the Elements in Pivot Row to Pivot Element, to make the pivot element 1. This row is the second step tableau's corresponding row.

To obtain second step tableau;

(iv) Multiply the new Pivot Row with a suitible number to make all other column elements 0 and add this row to other rows, obtain second step tableau rows.

Followin these steps; we have,

(i) -6 is the Most Negative Value.

(ii)

x_3	b	Ratio
-6	0	—
1	4	4:1 = 4
1	4	4:1 = 4
$\boxed{2}$	4	4:2 = 2
1	3	3:1 = 3

2 Minimum Positive Ratio

(3) S_3 → Pivot Row

→ Pivot element

(iii) Divide each element in pivot row to 2. New Row is;

New Basic, Replace S_3 with x_3

	Z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
S_3	$\frac{0}{2}=0$	$\frac{1}{2}=0{,}5$	$\frac{1}{2}=0{,}5$	$\frac{2}{2}=1$	$\frac{0}{2}=0$	$\frac{0}{2}=0$	$\frac{1}{2}=0{,}5$	$\frac{0}{2}=0$	$\frac{4}{2}=2$

→ we made Pivot element = 1

(iv) Make the Row operations; $6R_3' + R_0 = R_0'$

$-1R_3' + R_1 = R_1'$

$-1R_3' + R_2 = R_2'$

$-1R_3' + R_4 = R_4'$

New Rows (Second step tableau.

The second step tableau is;

	Basic	Z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	b
(0)	Z	1	0	-2	0	0	0	3	0	12
(1)	S_1	0	1,5	0,5	0	1	0	-0,5	0	2
(2)	S_2	0	0,5	1,5	0	0	1	-0,5	0	2
(3)	x_3	0	0,5	0,5	1	0	0	0,5	0	2
(4)	S_4	0	0,5	0,5	0	0	0	-0,5	1	1

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* As a final application, consider the problem of writing the LP problem from simplex tableau.

Example. Given the simplex tableau below, Express the problem as an LP model.

Basic	Z	x_1	x_2	x_3	s_1	s_2	s_3	b
Z	0	1	-1	-2	0	0	0	0
s_1	0	1	2	-1	1	0	0	20
s_2	0	-2	4	2	0	1	0	60
s_3	0	2	3	1	0	0	1	50

Answer; We have; $z + x_1 - x_2 - 2x_3 = 0$ or;

Maximize $z = -x_1 + x_2 + 2x_3$

subject to the constraints

$$x_1 + 2x_2 - x_3 \leq 20$$

$$-2x_1 + 4x_2 + 2x_3 \leq 60$$

$$2x_1 + 3x_2 + x_3 \leq 50$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

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